PHD

Arbitrage pricing model: A critical examination of its empirical applicability for the London Stock Exchange.

Diacogiannis, George P.

Award date:
1983

Awarding institution:
University of Bath

ARBITRAGE PRICING MODEL :

A Critical Examination of its Empirical Applicability for the London Stock Exchange.

Submitted by

GEORGE P. DIACOGIANNIS

For the Degree of

DOCTOR OF PHILOSOPHY

Of the University of Bath

1983

Διακογιάννης

ACKNOWLEDGEMENTS

I would like to take the opportunity to express my thanks most sincerely to my supervisor Dr. Mario Levis for his inspiring guidance, valuable suggestions and useful discussions throughout the preparation of this research.

I also wish to convey my appreciation to Professor C. Tomkins of Bath Universeity for his many helpful comments and advice.

Sincerest thanks are extended to Mr. G. Thanassoulas for his assistance in the computing aspects of this work.

Thanks are also due to the technical staff in the Computer Department of Bath University for their continuous help. I would particularly like to acknowledge the help of Mr. D. Clark.

The secretarial and library staff in Bath University are gratefully acknowledged, in particular Miss D. Stoddart.

In addition, I thank Mrs. J. Kitchin for the care she has taken and the speed in which she has completed the typing of this thesis.

Finally, I would like to express my deepest gratitude to my wife, my parents and my grandmother whose sacrifices made my research possible. I cannot say how much I owe to them.

A B S T R A C T

This research is concerned with the empirical verification of the assumptions necessary to ensure an unambiguous test of the arbitrage pricing model for the London Stock Exchange. More specifically, the purpose of this study is threefold :
First, to test the normality assumption regarding the distributions of security returns and the intertemporal stationarity assumption of the security mean returns and the covariance (correlation) matrix of security returns.
Second, to verify whether the number of common factors determining the security returns is the same across various groups of securities having different sizes and across different security groups having the same size.
Third, to test whether the number of common factors affecting the security returns remain unchanged across various time periods for the same group of securities and across various time periods for different groups of securities.

The research findings indicate that the distributions of security monthly returns are approximately normal and they are not intertemporally stationary. The correlation matrix of security returns seems to be stationary through time and thus the correlation matrix has to be used for the arbitrage pricing model's tests.

Furthermore the number of factors changes as the group size changes. Such results highlight that the

methodology used for testing the arbitrage pricing model is not the appropriate one, and previous tests of the arbitrage pricing model are not necessarily tests of the model. The arbitrage pricing model may be held, but the existing statitistical methodology does not provide an unambiguous test of the model for the London Stock Exchange.

Finally, the number of factors changes across various time periods for the same group of securities and for different security groups. These findings suggest that the security returns generating model of the arbitrage pricing theory cannot be used for making predictions. These results, however,do not constitute evidence against the arbitrage pricing model. The arbitrage pricing model may be held, but the present state of the statistical methodology cannot be utilized to provide an unambiguous test of the model for the London Stock Exchange.

C O N T E N T S

LIST OF TABLES

L I S T O F F I G U R E S

CHAPTER 1

GENERAL INTRODUCTION

1.1 Portfolio Theory : Some Basic Models

The modern portfolio theory began in 1952 with the pioneering work of Harry Markowitz. The portfolio theory describes how an investor should develop a set of efficient portfolios and then select the portfolio most suitable to his preference. A portfolio is said to be efficient if :
(i) No other portfolio with the same expected return can have lower variance of return.
(ii) No other portfolio with the same or lower variance of return can have higher expected return.

The original Markowitz model requires enormous computational resources to trace out the efficient frontier when a large number of securities are considered. In consequence index models have been developed to reduce the computational problems of the Markowitz formulation.

The capital market theory is built on the Markowitz portfolio theory. The objective of the capital market theory is to extend portfolio theory to a model that can be utilized to price all risky assets in the market. The final product was a model derived by Sharpe (1964), Lintner (1965) and Mossin (1966) known as the capital asset pricing model. The capital asset pricing model presents for securities or portfolios an equilibrium relationship between expected return and risk.

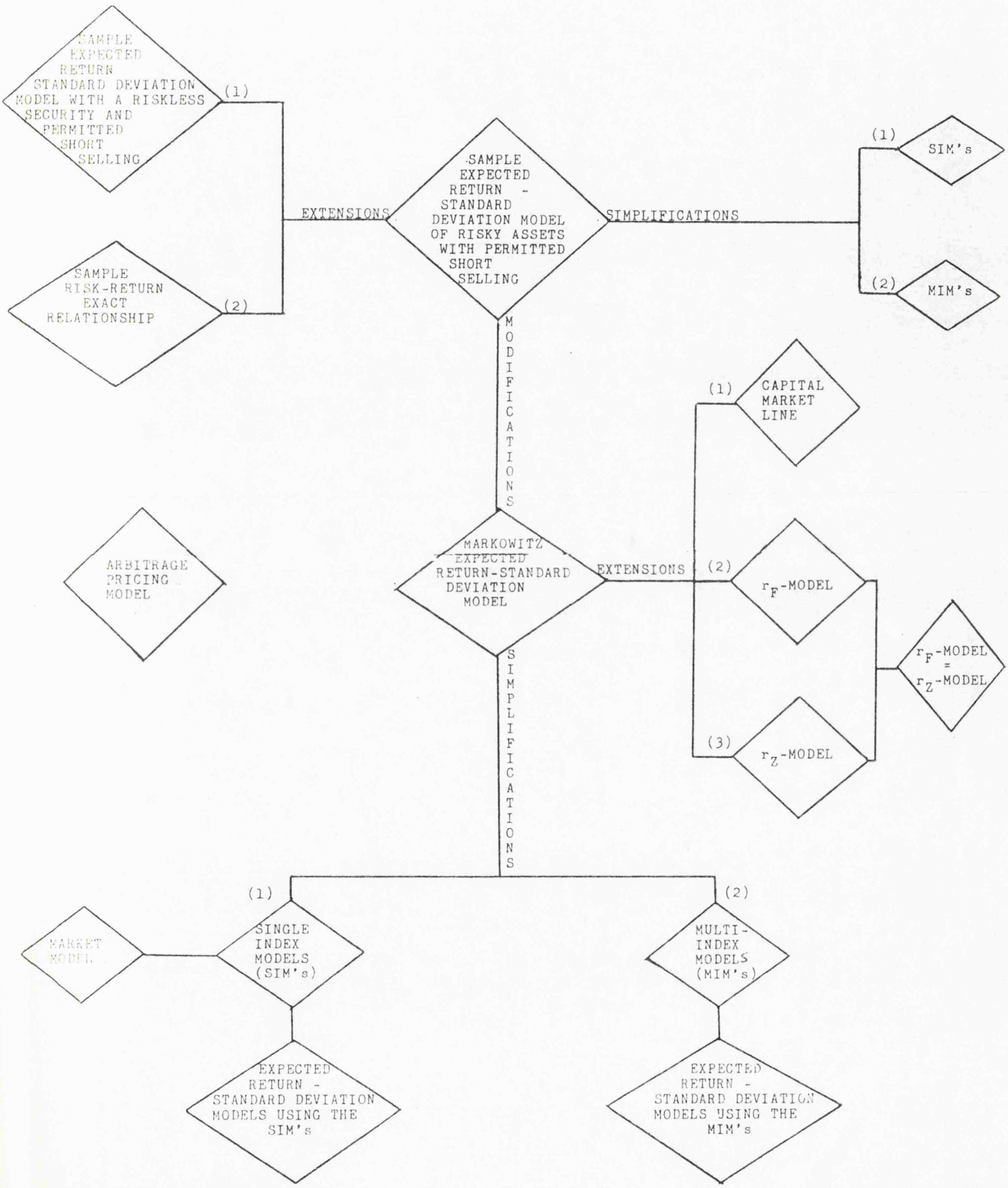
A considerable amount of effort has been expended in testing the capital asset pricing model. However, Roll (1977) pointed out that the major problem concerning the empirical examination of the capital asset pricing model is the problem of identifying the market portfolio. The model may be valid but it cannot be tested because the market portfolio is unobservable.

An alternative model for pricing risky assets is the arbitrage pricing model developed by Ross (1976, 1977). The arbitrage pricing model is not subject to the criticisms of the capital asset pricing model and thus it has received repeated attention in the literature.

This study is not concerned with linear models which have been based on three moments of investment return distributions - the expected return, the variance of return and the skewness of return (see Kraus and Litzenberger(1976) and Becker (1977)). Also the study does not deal with the extensions of the original equilibrium model produced by relaxing some of their initial assumptions (see Litzenberger and Ramaswany (1979), Levy (1978), Mayers D. (1972, 1973) and Stapleton and Subrahmanyam (1978)) .

Figure 1.1 presents a summary of the most important models in portfolio theory.

Figure 1.1 Some basic models of portfolio theory



1.2 The Objective of the Study

The Capital Asset Pricing Model (C.A.P.M.) is an elegant and simple model for pricing risky securities. Unfortunately the C.A.P.M. has been under strong criticism because the market portfolio has not been identified and thus tests of its empirical validity cannot be constructed.

An alternative approach to characterization of expected returns on risky securities is the Arbitrage Pricing Model (A.P.M.) proposed by Ross (1976, 1977).

Several empirical studies have been concluded that the A.P.M. can be verified empirically. Gerh (1978), Roll and Ross (1980), Chen (1981), Reinganum (1981), Johnson (1981) and Hughes (1982) provided some evidence of testing the A.P.M. These tests, however, are based on a number of empirical assumptions concerning the structure of data whose validity cannot be always guaranteed. Unfortunately the studies mentioned previously assumed that these empirical assumptions are met, and no special tests were made to verify the assumptions. Therefore these tests of the A.P.M. may be characterized as incomplete and so it cannot be inferred that the A.P.M. has been tested in an unambiguous fashion.

The objective of this study is to provide a first test of the assumptions necessary to ensure an unambiguous test of the A.P.M., for the London Stock Exchange.

More specifically, the objective of this study is threefold :
First, to test the normality assumption regarding the distributions of security returns and the intertemporal stationarity assumption of the security mean returns and the covariance (correlation) matrix of security returns.
Second, to verify if the number of factors which affect security returns remains the same across various groups of securities having different sizes and across different security groups having the same size.
Third, to test if the number of common factors having influence on security returns remains unchanged across various time periods for the same group of securities and across various time periods for different groups of securities.[1]

The A.P.M. alone provides no empirical hypothesis and thus the tests concerning its empirical validity require the ultilization of time series data and the employment of multivariate statistical techniques.

The empirical verification of these assumptions is very important, since initially it is examined whether security returns can be described by a single factor or a multi-factor linear model which is stationary through time. Furthermore, the number of factors determining the security returns is investigated. This investigation is of great importance because a large number of factors affecting the security returns would severely inhibit the practical applications of the model.

---

1. Since this work was started a number of studies have appeared touching on topics dealt with in this study (see Gibbons (1981), Johnson (1981), Kryzanowski and Chau (1982) and Dhrymes, Friend and Gultekin (1982)).

It is also important to examine empirically these assumptions, for if one or more of such assumptions are violated the usual tests for statistical inference are invalid. As a consequence the conclusions about the empirical validity of the A.P.M. will be highly suspect and hence the practical applications of the model will be questionable.

Therefore it is necessary before drawing any conclusion regarding the validity of the A.P.M. to verify empirically these assumptions. A violation of these assumptions does not constitute evidence against the A.P.M. - it simply shows that the multivariate statistical procedure employed to test the A.P.M. is not the appropriate procedure.

1.3 Limitations of the Study

In this empirical work there exist the following limitations:

(1) No attempt is made to determine the number of common factors which are "priced" .

(2) There is no attempt to identify the common factors.

1.4 An Overview of the Study

This study is organized into 12 chapters.

Chapter 2 summarizes the Markowitz's normative expected return-variance theory and it concentrates on the most important linear models built on the Markowitz concept.

Chapter 3 deals with the empirical evidence related to the linear models of portfolio theory and it highlights the criticisms of the well-known C.A.P.M.

Chapter 4 describes the A.P.M. and it proves such a model by using a weaker "no arbitrage" condition from that of Ross.

Chapter 5 is entirely devoted to the comparison of the C.A.P.M. and the A.P.M.

Chapter 6 reviews the empirical evidence of the A.P.M.

Chapter 7 is concerned with the applications of the factor analytic techniques to portfolio theory.

Chapter 8 presents the U.K. samples and the research methodology employed in this empirical work.

Chapter 9 examines the type of the joint probability distribution of security monthly returns and applies the appropriate methodology to test the intertemporal stationarity of the security mean returns, the covariance matrix of security returns and the correlation matrix of security returns.

Chapter 10 adopts the appropriate methodology to investigate empirically the relationship between the number of factors having influence on security returns and the group size being factored.

Chapter 11 employs the appropriate methodology to verify whether the number of factors which affect the security returns changes across various time periods.

Chapter 12 contains a summary of the conclusions of the study and gives some suggestions for further research.

CHAPTER 2

MARKOWITZ MODEL : SIMPLIFICATIONS AND EXTENSIONS

Three decades ago, Markowitz (1952) developed a theory which provides a framework of risk reduction through diversification. Markowitz showed that if risk is measured by the standard deviation of return and securities are not perfectly positively correlated then the investors can always reduce risk, while maintaining returns, by holding a portfolio of securities instead of holding a single security.

The original Markowitz model is extremely complex when applied in practical situations involving a large number of securities. Index models, therefore, have been developed to simplify its computational complexity .

The Markowitz's portfolio selection model can be generalized to a theory of equilibrium in capital markets. The most notable contribution to the theory of equilibrium in capital markets comes from the works of Sharpe (1964), Lintner (1965), Mossin (1966), Fama (1968) and Black (1972).

This chapter summarizes the Markowitz model and its simplifications and gives a brief overview of its major extensions.

2.1 The Markowitz Model

The modern theory for the intelligent selection of optimum portfolios under conditions of risk was developed by Markowitz (1952) and it is referred to as portfolio theory.

Under a number of assumptions concerning investors' behaviour Markowitz's formulation proceeds basically in three steps :

(A) Formation of the risk-return characteristics of N individual securities, where N is a finite positive integer.

(B) Computation of the efficient frontier of risky securities.

(C) Selection of an efficient portfolio that maximizes the investor's expected utility.

Markowitz's analysis has been couched in terms of the first two moments of return distributions - the expected return and the variance of returns. However, it pays no attention to the implications of assuming that security returns are random "drawings" from a normal distribution. These implications were extensively examined by Fama and Miller (1972) and Fama (1976).

2.2 Portfolio Risk and Security Risk

For an individual security or portfolio considered in isolation, an appropriate risk measure is the standard deviation of return.

Within a diversified portfolio, however, a different risk measure is important, that is the contribution of an individual security to the variance of the portfolio. Therefore within a diversified portfolio an appropriate measure of a security's risk is the covariance between the return on the security and the return on the portfolio (see Fama(1976,pp.58-62)).

2.3 The Single Index or Diagonal Model

A major disadvantage of the Markowitz model is that it requires a large number of calculations - pairwise covariances - when a high number of securities are available for consideration by the investor. To reduce the number of calculations, a model was suggested by Markowitz (1959, p. 100) and later developed by Sharpe (1963) that relates linearly the return on a security and the return on a market index. That is :

$$\tilde{R}_{it} = a_i + b_i \tilde{R}_{Mt} + \tilde{\varepsilon}_{it} \qquad (2.1)$$

where

$\tilde{R}_{it}$ = the (actual) t^{th} return on security i. The tilde "$\sim$" indicates that R_{it} is a random variable at the beginning of period t.

$\tilde{R}_{Mt}$ = the (actual) t^{th} return on the market index.

a_i = the expected return of the component of the security i's return that is independent of changes in the market index .

b_i = a measure of the responsiveness of the return on security i to changes in the return on the market index.

$\tilde{\varepsilon}_{it}$ = the disturbance term, with zero expected return and a constant variance. It is also assumed that $\tilde{\varepsilon}_{it}$ are uncorrelated with $\tilde{R}_{it}$ and the i^{th} security's ε's are independent with any other security's ε's.

Under this model the portfolio selection procedure can be achieved by following the same steps as in Markowitz's model. But this time the steps (A) and (B) require many fewer computations.

2.4 Multi-Index Models

The single index model assumes that the returns of securities are related only through a common market index. But one may choose a market index, say M, that has a low coefficient of determination (the coefficient of determination can be interpreted as the proportion of the variance of $\tilde{R}_{it}$ that can be attributed to the relationship between $\tilde{R}_{it}$ and $\tilde{R}_{Mt}$). In this case it would be preferable to introduce a multi-index model which might explain a high proportion of the security return's variability.

Cohen and Pogue (1967) presented the following two multi-index models:

(I) Multi-Index Model - The Covariance Form

This model requires the market to be segmented into m_1 security groups, where $m_1 = 1,2,\ldots,m$. Then it assumes the following linear relationship :

$$\tilde{R}_{it} = a_i + b_{i1}\tilde{R}_{I1t} + b_{i2}\tilde{R}_{I2t} + \ldots + b_{im}\tilde{R}_{Imt} + \tilde{e}_{it} \qquad (2.2)$$

where

$\tilde{R}_{it}$ = the return on security i in period t .

$\tilde{R}_{Im_1t}$ = the return on index m_1 in period t, $m_1 = 1,2,\ldots,m$.

a_i = the expected return of the component of the security i's return that is independent of changes in the indices.

b_{im_1} = a measure of the responsiveness of the return of security i to changes in the return on the index m_1 $m_1 = 1,2,\ldots,m$.

$\tilde{e}_{it}$ = the distrubance term, with zero expected return and a constant variance. It is also assumed that $\tilde{e}_{it}$ are uncorrelated with R_{Im_1t}, $m_1=1,2,\ldots,m$,and the i^{th} security's e's are independent with any other security's e's.

By assuming such a multi-index model, portfolio selection can proceed by following the same steps as in Markowitz formulation. Here also the steps (A) and (B) require many fewer computations.

(II) Multi-Index Model - The Diagonal Form

This model has the same basic structure as the covariance form. In addition, it assumes that there exists a linear relationship between each index and a general market index.

In this case also the portfolio selection procedure is much simpler that this of Markowitz.

Elton and Gruber (1981) proved that any multi-index model with correlated indices can be reduced to a multi-index model with orthogonal indices. They called such a model a general multi-index model. They also showed that the diagonal form of Cohen and Pogue's (1967) multi-index model is a special case of the general multi-index model (pp. 150-155).

2.5 The Market Model

The main disadvantage of the single index model is due to the assumption of independence among the disturbance terms of individual securities. It has been pointed out, however, by Fama (1968) that there is not consistency between this assumption and the use of the market portfolio as the factor affecting security returns. Fama proposed an index model with an independent variable the return of a common factor which affects the returns of all securities.

Furthermore, Jensen (1969, fn 24, p.178) pointed out that Fama's comments applied even though the return on the common factor is an average of all security returns.

Thus, another model having the same form as the single-index model and avoiding the assumption of independence among the disturbance terms is needed. The market model serves this requirement.

The market model embodies the following assumption :

(I) The joint distribution between any security's return and the return on the market portfolio is bivariate normal.

The market portfolio is a huge portfolio containing all the market risky securities - i.e. commodities, art objects, coins, shares, bonds, real estates, land, houses and anything else that can be given in terms of money - weighted according to their relative value in the market.

By means of assumption (I) the following linear relationship is implied for each security or portfolio :

$$\tilde{R}_{st} = a_s + b_s \tilde{R}_{Mt} + \tilde{\varepsilon}_{st} \qquad (2.3)$$

where

$\tilde{R}_{st}$ = the (actual) return on security or portfolio s during the period t. The tilde "$\sim$" indicates that R_{st} is a random variable at the beginning of period t.

$\tilde{R}_{Mt}$ = the (actual) return on the market portfolio during the period t.

$$b_s = \frac{\text{covariance between the returns on s and M}}{\text{variance of returns on M}} = \frac{\sigma_{sM}}{\sigma_M^2}$$

$a_s = r_s - b_s r_M$, with r_s, r_M be the one-period expected returns on s and M respectively.

$\tilde{\varepsilon}_{st}$ = the disturbance term with zero expected return and constant variance. $\tilde{\varepsilon}_{st}$ is also independent with the return on M.

Fama (1973, 1976) gives a complete discussion of the market model.

If M is a market proxy, and if the joint distribution between any security's return and the market proxy's return is bivariate normal, then one can produce a relationship analogous to that described by equation (2.3). In this case it can also be used a market proxy whose return is a weighted average of the security returns in the sample.

2.6 Single-Index Model versus Market Model

Below is presented a comparison between the single-index model and the market model.

(i) The single-index model assumes that security returns are related linearly to the market index's return and the specific return. (By specific return is meant that part of security's return that is not dependent on the market index's return). The market model assumes that the joint distribution between any security's return and the market portfolio's return is bivariate normal. This assumption in turn implies that security returns are related linearly to the market return and the specific return.

(ii) The single-index model assumes that the market portfolio's return and the disturbance term are uncorrelated. This, however, does not imply that they are independent. The market portfolio's return and the disturbance term are independent if the following two conditions are satisfied :

(a) The joint distribution between any security's disturbance term and the return on the market portfolio is bivariate normal.

(b) There is no correlation between the market portfolio's return and the disturbance term.

The market model's assumption (I) implies that the market portfolio's return and the disturbance term are independent and hence uncorrelated.

(iii) The single-index model assumes that there are not interdependencies among the disturbance terms of different securities.
In the market model, always the disturbance terms are related linearly.

(iv) In the single-index model the independent variable's return can be neither the market portfolio's return nor a weighted average of all security returns.
In the market model the independent variable's return can be the market portfolio's return.

2.7 Capital Market Theory

The Capital Market Theory (C.M.T.) is one of the major extensions of the Markowitz two-parameter portfolio analysis model.

The C.M.T. describes how capital assets are priced in the market if all investors are Markowitz diversifiers and if there are equilibrium conditions in the market.

The main contribution to the C.M.T. is due to Sharpe (1964), Lintner (1965), Mossin (1966), Fama (1968) and Black (1972). Among the questions investigated by these authors were the following :

(A) What is the equilibrium relationship between return and risk for efficient portfolios ?

(B) What is the equilibrium relationship between return and risk for individual securities or inefficient portfolios ?

(C) What is the appropriate measure of risk for individual securities implied by the portfolio selection in the context of market equilibrium?

2.8 The Capital Market Line and the r_F-Model

By introducing a set of simplified assumptions, Sharpe, Lintner and Mossin have shown that the expected return-standard deviation efficient frontier of all investors in the market could be described by only two funds :

the market portfolio containing only risky securities and the riskless security. This efficient frontier, called <u>the Capital Market Line</u> (C.M.L.), presents a linear ex-ante equilibrium relationship between return and risk for efficient portfolios only. Moreover Sharpe, Lintner and Mossin derived a linear ex-ante equilibrium relationship between return and risk for securities or portfolios (efficient or not) called <u>the r_F-model</u>.

(I) <u>Assumptions Underlying the Capital Market Line and the r_F-Model</u>

Let us denote by Ω the population of all risky securities in the market place. The C.M.L. and the r_F-model depend on the following assumptions :

(A) <u>Assumptions about investor behaviour</u>

(1) Investors assume that the joint distribution on the single period security returns can be well approximated by a multivariate normal distribution, or their single period utility function of terminal wealth is a quadratic approximation.

(2) Investors prefer more expected portfolio return to less.

(3) Investors are risk averse.

(4) All investors have in common a single period investment horizon and identical expectations about the distributions of the security returns at the end of this horizon.

(5) Investors are expected utility of terminal wealth maximizers.

(B) Assumptions about the market

(6) Short sales of risky securities are not permitted.

(7) There exists a risk-free rate security such that investors may borrow or lend any amount at the risk-free rate.

(8) No inflation and no charge in the level of the risk-free rate exist.

(9) The capital market is perfect - that is :

(a) There are no transaction costs.

(b) Securities are infinitely divisible.

(c) Information is costless and available to every investor.

(d) No single buyer or seller of securities is large enough to affect their market price.

(10) There are no taxes.

(11) Security markets are in equilibrium.

(12) All securities are marketable.

(C) Mathematical assumption

(13) The covariance matrix of security returns is not singular.

(II) The Equation of the Capital Market Line

In view of the above assumptions, it can be shown that in equilibrium the expected return on an efficient portfolio is related linearly to its risk as follows :

$$r_q = r_F + \frac{r_M - r_F}{\sigma_M} \sigma_q \tag{2.4}$$

where

r_q = the equilibrium one-period expected return on an efficient portfolio.

r_M = the equilibrium one-period expected return on the market portfolio.

r_F = the one-period rate of return on the riskless security.

σ_q, σ_M = the standard deviations of q and M, respectively.

(see figure 2.1).

(III) The Equation of the r_F-Model

Making use of the previously mentioned assumptions, it is possible to prove theoretically the efficiency of the market portfolio. The efficiency of the market portfolio is then a necessary condition for the validity of the ex-ante equilibrium linearity relationship between the return on a security or portfolio (efficient or not) and its risk in the market portfolio. That is the market portfolio's efficiency implies the following exact linear relationship for each s :

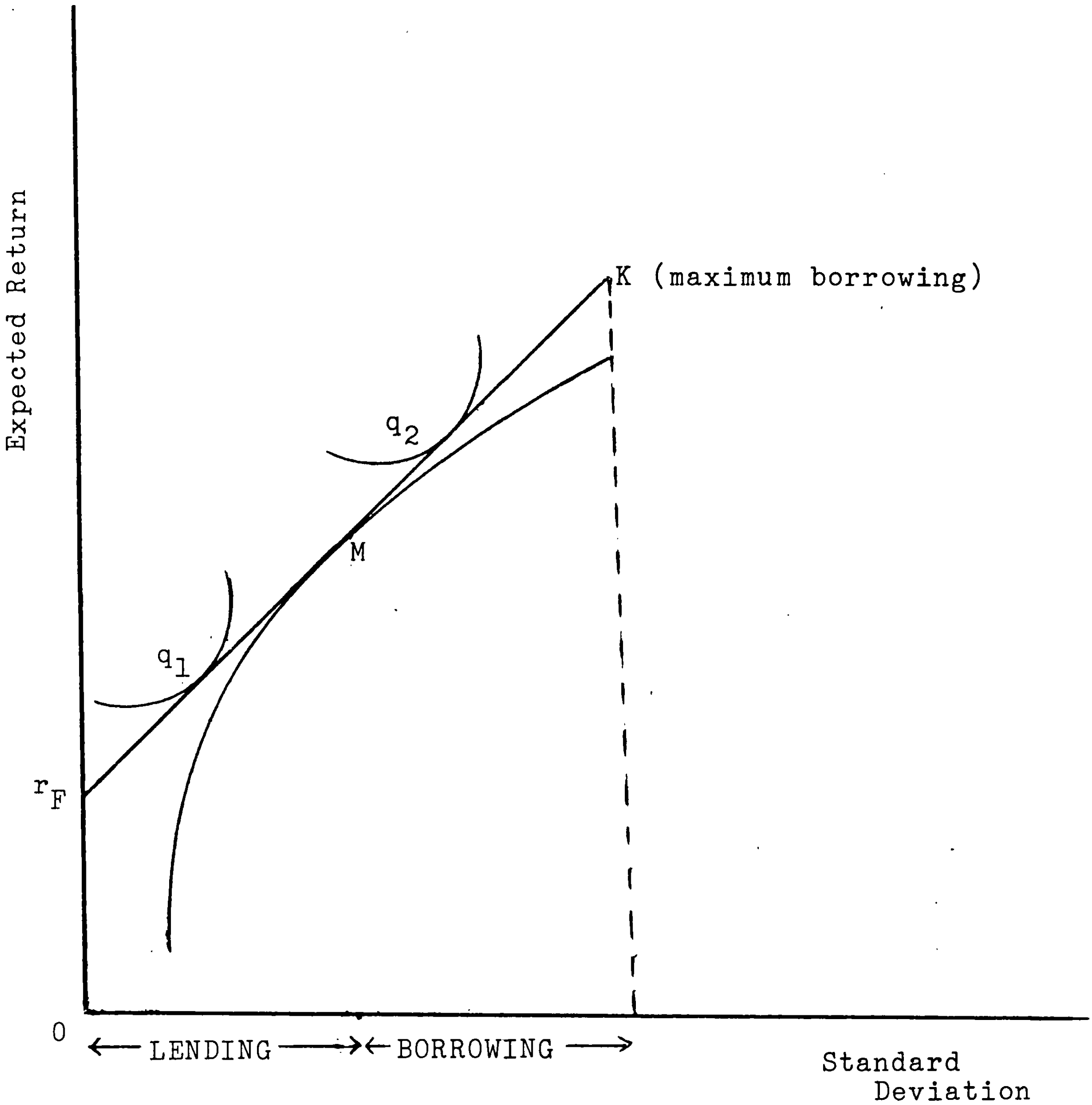
$$r_s = r_F + (r_M - r_F) b_s \tag{2.5}$$

where

r_s = the equilibrium one-period expected return on a security or portfolio (efficient or not).

r_M = the equilibrium one-period expected return on the market portfolio.

Figure 2.1 The capital market line and indifference curves .



$r_F K$ = capital market line.
q_1 = optimal portfolio for strongly risk-averse investor.
q_2 = optimal portfolio for middly risk-averse investor.

b_{sM} = the relative risk of s in M .

2.9 The r_Z-Model

Black (1972) considered the case where investors cannot borrow or lend at the risk-free rate. By taking into account a set of simplified assumptions, he arrived at the conclusion that the expected return-standard deviation efficient frontier of all investors in the market could only be described by two funds : the market portfolio and another minimum variance portfolio chosen to be orthogonal to the market portfolio. Black also produced an ex-ante equilibrium relationship among return and risk of securities or portfolios, irrespective of whether they are efficient or not, called the r_Z-model .

(I) Assumptions Underlying the r_Z-Model

Let Ω be the population of all risky securities in the market place. The r_Z-model rests upon the assumptions (1) - (5) and (9) - (13) of the r_F-model and on the following assumption :

(1) There is unrestricted short selling of risky securities.

(II) The Equation of the r_Z-Model

With the aid of the r_Z-model's assumptions the efficiency of the market portfolio can be proved theoretically. The efficiency of the market portfolio is then a necessary condition for the validation of the ex-ante linear equilibrium relationship between the return on a security or portfolio (efficient or not) and its risk in the market portfolio. It follows that, for each security or portfolio the following equation is a consequence of the efficiency of the market portfolio :

$$r_s = r_{ZM} + (r_M - r_{ZM}) b_s \tag{2.6}$$

where

r_{ZM} = the equilibrium one-period expected return on the minimum standard deviation portfolio, whose return is uncorrelated with the return of the market portfolio.

The r_F-model and the r_Z-model are the two versions of the well known C.A.P.M.

2.10 r_F-Model versus r_Z-Model

A simple comparison between the r_F-model and the r_Z-model shows that :

(1) Each model is based on a set of assumptions made about the investors and the market. Both sets contain the same assumptions except :

(a) The r_F-model assumes the existence of a riskless security, where investors can borrow or lend any amount of money.
The r_Z-model does not assume the existence of the riskless security.

(b) The r_F-model does not require short selling of risky securities.

The r_Z-model assumes unrestricted short selling of risky securities.

(2) The assumptions of the r_F-model imply the following :

(a) All investors must perceive the same expected return - standard deviation efficient frontier of risky securities.

(b) In market equilibrium only one portfolio of risky securities is considered. This portfolio is determined by the point of tangency between the efficient frontier of risky securities and the straight line passing through the riskless rate of interest. Such portfolio is the market portfolio.

The implications of the r_Z-model's assumptions are stated as follows :

(a) In market equilibrium the market portfolio is a convex linear combination of expected return-standard deviation efficient portfolios chosen by investors,where each investor's portfolio is weigthed by the ratio of his invested wealth to the total invested wealth of all investors.

(b) A portfolio is a minimum standard deviation portfolio if, and only if, it is a convex linear conbination of m other distinct minimum standard deviation portfolios,where $m \geqslant 2$. The market portfolio is a convex linear combination of m other distinct efficient portfolios (hence minimum standard deviation portfolios).So the market portfolio is a minimum standard deviation portfolio. But the market portfolio's weights are positive real numbers and thus the market portfolio is an expected return-standard deviation efficient.

Therefore the assumptions of each model imply the theoretical location of the market portfolio on the efficient frontier of risky securities. However, when the r_F-model assumptions are taken into account, the method of proving theoretically the efficiency of the market portfolio is different from the method used when the r_Z-model's assumptions are taken into consideration.

(3) In theory, both models are discussed in terms of populations of securities and both rely on the efficiency of the market portfolio.

(4) Both models are equilibrium models.
The r_F-model explains the equilibrium linear relationship between expected return and risk of :

(a) Each risky security in the market.

(b) Each expected return-standard deviation efficient portfolio . These portfolios are linear combinations of the market portfolio and the riskless security. These portfolios have returns which are perfectly positively correlated with the market portfolio's return.

(c) Each expected return-standard deviation inefficient portfolio. Each of these portfolios is a linear combination of the riskless security and a minimum standard deviation portfolio generated by risky securities. Each portfolio is defined by a vector of positive weights and it has expected return less than the expected return on the market portfolio.

On the other hand the r_Z-model explains the equilibrium linear relationship between expected return and the risk of :

(a) Each risky security in the market.

(b) Each minimum standard deviation portfolio. These portfolios are linear combinations of the market portfolio and its minimum standard deviation orthogonal portfolio. None of these portfolios has a return that is perfectly correlated with the return on the market portfolio.

(c) Each expected return-standard deviation portfolio which is not a member of the minimum standard deviation portfolio set of risky securities.

(5) The optimal portfolio for a risk-averse investor must be expected return-standard deviation efficient. Therefore the r_F-model's assumptions imply that risk-averse investors hold portfolios whose returns are perfectly positively correlated, whereas the r_Z-model's assumptions imply that risk-averse investors hold portfolios whose returns are not perfectly correlated.

(6) The r_F-model in the expected return-beta plane is described by a straight line. This straight line has a positive slope and it intersects the expected return axis at r_F.
The r_Z-model in the expected return-beta plane is also represented by a straight line. Such a line has a positive slope and it intersects the expected return axis at r_Z.

If the assumption (6) of the r_F-model is substituted with the short selling assumption of the r_Z-model it can be concluded that $r_F = r_{ZM}$. Only in this case it can be produced an r_F-model which is identical to the r_Z-model.

2.11 Capital Asset Pricing Model versus Market Model

Comparing the C.A.P.M.and the market model one observes :

(1) The C.A.P.M. is an ex-ante model which highlights a cross-sectional equilibrium relationship between expected return and risk.
The market model is not an equilibrium model. It is mainly concerned with explaining the return generating process of securities. Consequently it is an ex-post model.
(2) The C.A.P.M. is based on the multivariate normality assumption which is consistent with the joint normality assumption underlying the market model.
(3) The C.A.P.M. has price of risk implications : that is it indicates the extra return that can be gained by increasing the measure of risk (beta) on a security or portfolio (efficient or not) by one unit.
The market model has no price of risk implications.

Hence it may be inferred that the C.A.P.M. is generally distinct from the market model. Only in an efficient market, when all the regression assumptions are satisfied and the underlying process is stationary through time, the ex-post form of the C.A.P.M. is the same as the market model's form. (see section 3.2.2).

2.12 Beta as a Measure of Risk

Assuming that the joint probability distribution of security returns is multivariate normal, it may be inferred that the joint distribution between any security's return and the market portfolio's return is bivariate normal. In this case the equation of the market model is valid. The coefficient b_s of the market model is equal to σ_{sM}/σ_M^2 and it measures the risk of s in M relative to the risk of the market portfolio M. So b_s is called the relative risk of s in the market portfolio.

There are three main justifications behind the use of beta as a measure of risk. These are :

(1) By means of the market model, the variability of future returns on a security s (i.e. its riskiness) can be broken into two components:

(a) A systematic component generated by the market and thus called systematic risk.

(b) An unsystematic component generated by the ε's and hence called unsystematic risk.

When securities are combined in a well diversified portfolio, the unsystematic components become relatively unimportant, leaving the component generated by the market as the major contributor to the portfolio risk. Therefore, the risk of a well diversified portfolio will be comprised of the systematic risk only. Since the risk of the market portfolio, as measured by the standard deviation of return, is a constant with respect to all securities, b_s measures the contribution of the security's s risk to the total riskiness of the diversified portfolio.

(2) Beta is a useful measure of risk for an individual security and it can be utilized when is added a new security to a portfolio. In fact :

(a) If $b_s < 1$, then s can be used to lower the variance of M.

(b) If $b_s > 1$, then s cannot be used to lower the variance of M. (Babcock (1972))

(3) The r_F-model can be written as

$$r_s - r_F = b_s (r_M - r_F) \tag{2.7}$$

where

$r_s - r_F$ = the expected excess return on the s^{th} security.

$r_M - r_F$ = the expected excess return on the market portfolio.

The last equation shows that in equilibrium b_s is the constant of proportionality between the expected excess return on the s^{th} security and the expected excess return on the market portfolio (Blume (1971)).

The same conclusion also holds for the r_Z-model.

2.13 The Sharpe Multi-Beta Model

Sharpe (1977) gave to the C.A.P.M. a multi-beta interpretation. He assumed that the market portfolio is generated by N risky securities and that the market portfolio in turn is used to generate k portfolios. Then he argued that the relative risk of a security in the market portfolio can be represented as a convex linear combination of the quantities b_{sj}/b_{Mj}. That is

$$b_{sM} = \sum_{j=1}^{k} u_j \frac{b_{sj}}{b_{Mj}} \tag{2.8}$$

where

$s = 1,2,\ldots,N$.

$\sum_{j=1}^{k} u_j = 1$.

$$b_{sj} \equiv \frac{Cov(R_s\ R_j)}{\sigma^2(R_j)} = \frac{\text{Covariance of the security s's return with that of the portfolio j}}{\text{Variance of the return on the portfolio j.}}$$

$$b_{Mj} \equiv \frac{Cov(R_M\ R_j)}{\sigma^2(R_j)} = \frac{\text{Covariance of the market portfolio's return with that of the portfolio j.}}{\text{Variance of the return on the portfolio s.}}$$

$$R_M = \sum_{j=1}^{k} W_j R_j \ . \tag{2.9}$$

$W_j \equiv$ the total proportion of the market value invested in portfolio j.

R_j = the rate of return on portfolio j.

Next Sharpe pointed out that factors as inflation, taxation considerations, etc., affect the returns on the N securities and thus will also affect the return on the market portfolio. Therefore he assumed that the security returns are determined by f observable factors and that the return on the market portfolio can be expressed as :

$$\tilde{R}_M = \sum_{g=1}^{f} b_{Mg}\tilde{F}_g + \tilde{C}_M$$

where

$F_1, F_2, \ldots, F_g$ = the returns on the observable factors affecting the security returns.

b_{Mg} = the sensitivity of the market portfolio's return to fluctuations in the common factor g .

$\tilde{C}_M$ = the (random) difference between $\tilde{R}_M$ and the proportion of $\tilde{R}_M$ attributable to the factors.

Then by taking into account the assumption that the variance of C_M approaches to zero he asserted that the relative risk of a security in the market portfolio is (approximately) related to b_{Mg} (= the relative risk of M in the portfolio g).

In this case the results obtained earlier can now be applied. By interpreting R_j in equation (2.9) as the value of factor j and W_j as the sensivity of R_M to a change in factor j, then equation (2.8) states that the relative risk of the security in the market portfolio is related to b_{sj} (= the sensitivity of R_s to a change in factor j).

Consequently the r_F-model can be rewritten as :

$$r_i = r_F + (r_M - r_F) \sum_{j=1}^{k} \left(\frac{u_j}{b_{Mj}} \right) b_{sj} \tag{2.10}$$

Equation (2.10) represents a "multi-beta" interpretation of the r_F-model. The identification of the factors affecting the return of the market portfolio can only be achieved if one identifies the market portfolio itself. However, it will be explained in section 3.3 that portfolio models based upon the market portfolio's identification reveal the lack of empirical validity.

2.14 Conclusions

Markowitz was the first who developed a theory which discusses the rules for the systematic selection of optimum portfolios under conditions of risk.

To facilitate practical application to the original Markowitz model, Markowitz suggested and Sharpe developed the single-index model. The single-index model assumes that the security returns are linearly related only to the return of a common factor to all securities.

Multi-index models introduce beyond the market index some extra indices in the hope of capturing further information.

The market model has the same form as the single-index model, but it avoids one very strong assumption of the single-index model.

The C.M.T. is the major extension which is built on the Markowitz portfolio theory. The C.M.L. is an ex-ante equilibrium relationship that expresses linearly return and risk of efficient portfolios only. The r_F-model and the r_Z-model are two distinct ex-ante equilibrium relationships. Each one relates linearly return and risk for securities or portfolios (efficient or not). Both are generally distinct from the market model.

A security's or portfolio's risk, in isolation, is measured by the standard deviation of return. The risk of an efficient portfolio is also measured by the standard deviation of return. A security's risk within a diversified portfolio is measured by the covariance between the return on the security and the return on the portfolio. Finally, the beta coefficient measures the systematic risk of a security or portfolio (efficient or not) relative to the risk of the market portfolio.

CHAPTER 3

PORTFOLIO SELECTION MODELS: THE EMPIRICAL EVIDENCE

Chapter 2 described the Markowitz model, its simplifications and some of its extensions. The purpose of this chapter is to review the most important empirical studies of these portfolio selection models.

This chapter begins with a brief review of the empirical evidence concerning the original Markowitz model and its simplifications. Then it provides a summing up of the statistical methodology used in an attempt to test the C.A.P.M.'s versions. Finally, it discusses the criticisms of the C.A.P.M., especially those stated by Roll (1977) and points out the differences between the C.A.P.M. and the sample risk-return exact linear relationship.

3.1 Empirical Tests of the Markowitz Model and its Simplifications

The empirical studies concerned with the expected return-standard deviation Markowitz model and its simplifications can be divided into two groups.

(I) Those which tested the model as proposed by Markowitz. A study of this type was performed by Farrar (1967).

(II) Those which tested several index models which developed to simplify the inputs of the original Markowitz model. Empirical work in this area was offered by King (1966), Mayers (1973), Fertuck (1975), Cohen and Pogue (1967), Elton and Gruber (1973), Farrel (1974), Wallingford (1967), Alexander (1978, 1977), Aber (1976) and Livingston (1977).

The empirical studies of the second group can be divided into the following subgroups :

(i) Those which examined if industry effects are a significant determinant of security returns (King, Mayers and Fertuck).

(ii) Those which compared directly single-index models with multi-index models (Cohen and Pogue, Elton and Gruber, Farrel, Wallingford, Alexander, Aber and Livingston).

A brief review of these empirical investigations now follows:

Farrar, (1967) examined if actual portfolios of mutual funds approximate to optimal portfolios constructed as proposed by Markowitz. By taking into account a population made from industries and asset groupings and employing principal component analysis techniques, he formed a revised population consisting of eleven variables. As a next step he used the population of the ten variables and he computed the efficient frontier of optimal portfolios by solving a number of quadratic programming problems. He also made use of a sample of mutual funds and he compared their risk-return combination against his efficient frontier. His conclusions can be summarized as follows :

(1) Funds tended to cluster into groups.

(2) Funds were located close to the efficient frontier.

(3) Funds that claimed to be risky were located close to the high-risk efficient portfolios, while balanced funds were located close to the low-risk efficient portfolios.

King (1966) was the first who questioned the existence of important covariance between security returns beyond those attributable to an overall market factor. He attempted to determine how much of the variability of security returns was attributable to overall market fluctuations, and how much could be attributed to industry factors. He used factor analysis techniques and he concluded that on average for his overall sample period, 52 per cent of the total variation in stock's returns was accounted for by a market factor. After removing the market factor and employing cluster analysis he deduced that, on average over his overall sample period, 10 per cent of the variance of return was explained by industry factors.

King's results indicated that industry factors are important factors affecting security returns. Moreover his findings indicated that security returns are also affected by other factors beyond the market factor and his industry factors.

Mayers (1973) subsequently demonstrated that King's results overstate the role of industry factors in the market. Using principal component analysis techniques he reported empirical evidence that confirmed the findings of King regarding market influence and the importance of industry analysis. However, he inferred that there are industries different from those studied by King, which did not have significant effect on returns as the ones that King studied.

Fertuck (1975) attempted to resolve the contradictions between King's and Mayer's results. He used three digit standard industrial classification (S.I.C.) industry indices to account for variance in monthly returns. His findings showed that an industry index can be used to explain 11.5 per cent of the variance of the return. He argued. however, that these results cannot be generalised to all industries. To support his argument he showed that there are industries for which the S.I.C. codes explain only 1.4 per cent of the variance of return. Fertuck inferred that :

> "It is necessary to be very careful when deciding whether to use an industry index to remove systematic movements. In some industries, the industry effect is trivial and can be safely ignored. In others, it can be as large as a third of the market effect" (p.847).

According to the studies and King, Mayers and Fertuck, it is reasonable to expect that a multi-index model can be used to simplify the computational requirements of the original Markowitz model. In view of these studies, however, no conclusions can be drawn regarding the importance of industry factors, since King's findings are not supported by the studies of Mayers and Fertuck.

Cohen and Pogue (1967) made a direct comparison of efficient portfolios generated by a single-index model and a multi-index model ; they found that the former produces more efficient portfolios than the latter. Cohen and Pogue attributed this result to the fact that only common stocks were used and that their high intercorrelations made the group more amenable to a single-index model.

Elton and Gruber (1973) also constructed a study for the comparison of a single-index model and a multi-index model. Their findings were similar to those of Cohen and Pogue : their single-index model dominated their multi-index model in approximating the Markowitz model's efficient frontier.

Farrel's (1974) aim was to form homogeneous groups and to use them as inputs to a multi-index model. He pointed out that the results of Cohen and Pogue were due to the high correlation between the indices used as inputs to their multi-index model. Hence the utilization of homogeneous groups as inputs to the multi-index model is necessary. Farrel tested for extra effects in common stock returns related to growth, cyclical and stable groups. He separated stocks into groups based on the correlation in their residual returns and he found that four homogeneous groups emerged from his analysis : growth, cylical, stable and oil. Regression analysis results indicated that these stock groupings explained on average 14 percent of the variance of returns on stocks in the sample. Farrel concluded that it is necessary to add a fourth factor to those suggested by King as explaining the variance of returns on a common stock. Farrel also examined the residual correlation matrices produced by his single-index and four-index models. He found that his single-index model produced a correlation matrix with significant entries while his four-index model did not. Hence he deduced that his multi-index model does a better job of approximating the Markowitz model than the single-index model.

Wallingford (1967) performed a study similar to Cohen and Pogue. His results contradicted those of Cohen and Pogue - i.e. two-index model generated efficient portfolios that dominated those obtained with a single-index model.

Alexander (1978) tried to locate the main reason for the conflicting results produced by Cohen and Pogue and Wallingford. He documented results showing the superiority of the single-index model to the multi-index model. His results were in contrast to the results of Wallingford. He attributed this discrepancy to the choice of the indices. He noted that, when he used indices computed from the sample itself, the single-index model did not outperform the multi-index models. So his results lent support to the Cohen and Pogue conclusions.

Alexander's (1977) study was concerned firstly with forming efficient portfolios and secondly with comparing single and multi-index models when heterogeneous securities (i.e. common stocks, preferred stocks, corporate bonds and U.S. Government bonds) were taken into consideration. His results made him deduce that his multi-index model generated more efficient portfolios than those of his single-index model.

Aber (1976) compared a single-index model, an industry-based multi-index model and two non-industry-based multi-index models.

In his study he examined the correlations among the residuals for each model. On the base of his evidence he concluded that the multi-index models are superior to the single-index model, since the former produced independent residual terms, while the latter produced dependent residual terms. In addition, he inferred that his industry-based multi-index model was outperformed by his non-industry-based multi-index models.

Finally, Livingston (1977) examined the following implication of the single-index model:

> The correlation between the returns on any two securities after removing the market effect is zero.

He pointed out that the same implication was examined by other researchers, but their results were conflicting. In his study Livingston showed the superiority of the regression techniques to the factor analysis techniques. His reported evidence can be summarized as follows :

(1) There exist correlations between security returns after the market factor is removed.

(2) Diversification can be achieved by selecting securities from different industries.

(3) Multi-index models, having as inputs non-orthogonal industry indices, explain more fully the intercorrelations among securities than a single-index model.

From the results presented previously it may be concluded that there is not an unambiguous answer concerning the performance of the multi-index models relative to the single-index models. The most surprising outcome is the ability of the single-index model to outperform some multi-index models. Although every study has provided some justification concerning this surprising outcome, no special tests have been made to confirm that such an outcome is not due to the violation of the basic assumptions of the statistical method used.

3.2 Empirical Tests of the Capital Asset Pricing Model's Versions

This section begins with a series of testable implications regarding the C.A.P.M.'s versions.

3.2.1 Testable Implications of the Capital Asset Pricing Model's Versions

Each version of the C.A.P.M. has the following testable implications :

(1) The relationship between the expected return on a security or portfolio is linear to its relative risk in the market portfolio.

(2) A security's or portfolio's relative risk in the market portfolio is the only measure of risk that affects its expected return.

(3) In a market of risk-averse investors, the relationship between a security's or portfolio's expected return and its relative risk in the market portfolio is positive.

3.2.2 Testing Ex-ante Models with Ex-post Data

Since ex-ante measures generally cannot be observed, ex-ante models are tested by using ex-post data. There are two main points to be considered when ex-post data are used to test ex-ante models. These are :

(1) The assumption of the market efficiency. In efficient markets, the differences between expectations and realizations should average out to zero over reasonably long periods of time. This assumption implies that, over long periods of time, ex-post magnitudes can be used as proxies for ex-ante magnitudes.

(2) Since ex-ante models are always cast in terms of expectations, for testing purposes they must be transformed to ex-post relationships. For each version of the C.A.P.M., this can be achieved if a model is specified that relates ex-post returns to ex-ante returns. Such a model can be derived with the help of the market model, which is an ex-post model. It is assumed that the joint distribution of security returns is multivariate normal and stationary through time. Then the market model holds in each period (Fama (1976), Ch. 3). That is :

$$\tilde{R}_{st} = a_s + b_s\tilde{R}_{Mt} + \tilde{\varepsilon}_{st} \qquad (3.1)$$

where

$s = 1,2,\ldots,N$.

$t = 1,2,\ldots,T$.

$E(\tilde{\varepsilon}_{st}) = 0$ for each t .

$E(\tilde{\varepsilon}_{st}\tilde{R}_{Mt}) = 0$ for each t .

$E(\tilde{\varepsilon}^2_{st}) = \sigma^2_{\varepsilon_s}$ for each t .

$E(\tilde{\varepsilon}_{st}\,\tilde{\varepsilon}_{st+1}) = 0$ for each t .

E = the expectation operator.

The use of ex-post data implies the substitution of an ex-ante distribution by an ex-post distribution. Thus it has to be ensured that the ex-post and ex-ante returns on securities follow distributions of the same type. More precisely the r_F-model requires multivariate normality. This is perfectly consistent with the joint normality assumption on which the market model is heavily based.

Equation (3.1) implies that the expected return on s from period t to period t+1 is given by :

$$r_{st} = a_s + b_s r_{Mt} \tag{3.2}$$

where

r_{Mt} = the expected return on the market portfolio from period t to period t+1 .

Subtracting equation (3.2) from equation (3.1) and rearranging terms it follows that :

$$\tilde{R}_{st} = r_{st} + b_s(\tilde{R}_{Mt} - r_{Mt}) + \tilde{\varepsilon}_{st} \tag{3.3}$$

Equation (3.3) relates the t^{th} period realized return on s with its next period expected return. Expressed differently equation (3.3) relates ex-post measures with ex-ante measures. If the joint distribution of securities' returns is stationary through time, it can be assumed that the following relationship holds across securities in every period:

$$r_{st} = R_{Ft} + (r_{Mt} - R_{Ft})b_s \tag{3.4}$$

where

$s = 1,2,\ldots,N$.

$t = 1,2,\ldots,T$.

If the joint distribution of R_{st} and R_{Mt} is bivariate normal, then the beta coefficient produced by the market model is identical to the beta coefficient of the C.A.P.M. (Fama (1973)). Thus substituting equation (3.4) into equation (3.3) and simplifying, one takes

$$\tilde{R}_{st} = R_{Ft} + b_s\,(\tilde{R}_{Mt} - R_{Ft}) + \tilde{\varepsilon}_{st} \qquad (3.5)$$

Equation (3.5) is the ex-post form of the r_F-model. The non-autocorrelation assumption of the market model implies that over time the $\tilde{\varepsilon}_{st}$'s of a security add up to a value close to zero. Hence equation (3.5) can be summed over T and averaged to derive

$$\overline{R}_s = \overline{R}_F + b_s(\overline{R}_M - \overline{R}_F) \qquad (3.6)$$

where the bars indicate average returns over T .

Similarly one can produce the ex-post form of the r_Z-model. That is

$$\tilde{R}_{st} = \tilde{R}_{Zt} + b_s(\tilde{R}_{Mt} - \tilde{R}_{Zt}) + \tilde{\varepsilon}_{st} \qquad (3.7)$$

and thus to take

$$\overline{R}_s = \overline{R}_Z + b_s(\overline{R}_M - \overline{R}_Z) \qquad (3.8)$$

3.2.3 Statistical Methods used to Test the Capital Asset Pricing Model's Versions and Conclusions of the Tests

There have been several attempts to test the implications of the r_F-model or/and the r_Z-model. Studies by Lintner (in Douglas (1968), p.36), Jacob (1971), Miller and Scholes (1972), Black, Jenson and Scholes (1972), Blume and Friend (1973) and Fama and MacBeth (1973) constitute some of the more rigorous empirical tests.

In each of these studies a sample of securities with observed returns over a specific sample period has been utilized and a test of the implications of the r_F-model or/and the r_Z-model has been carried out. A brief of the statistical methods used in these empirical tests is provided below.

To examine empirically the first implication concerning the return-beta linearity the following two testing methodologies have been adopted :

(1) A two stage cross-section methodology.

(2) A time series methodology.

The two stage cross-section methodology followed a three-step procedure:

STEP 1 For each security or portfolio in the sample the beta coefficient was estimated.

STEP 2 Cross-sectional regressions between security or portfolio average realized returns and betas, or between security or portfolio realized returns and betas were run.

STEP 3 The coefficients of the cross-sectional regressions were compared with the risk free rate (the zero beta portfolio's mean return) and the mean excess on the market index respectively.

This methodology was adopted by Jacob (1971), Black, Jenson and Scholes (B.J.S.) (1972), Blume and Friend (B.F.) (1973) and Fama and MacBeth (F.M.) (1973).

Jacob for her tests used securities, while B.J.S., B.F. and F.M. aggregated securities into portfolios and they used portfolios for testing purposes. The major reason for working with portfolios rather than individual securities is to reduce the errors in estimates of betas so that consistent estimates may be obtained from the cross-sectional regression. This is true because, by combining securities that are less than perfectly positively correlated, diversification can be achieved. As a consequence, the specific risk on portfolios is smaller than the specific risk on individual securities. (By specific risk is meant that part of a security's or portfolio's total risk that cen be diversified away). Hence the errors in portfolio betas are smaller than those of individual securities.

Jacob and B.J.S. used the ex-post form of the r_F-model to estimate security betas, while B.F. and F.M. used the market model. The ex-post form of the r_F-model was utilized to estimate betas, because Roll (1969) demonstrated that, if the risk free rate fluctuates over time and if it is correlated with the return on the market portfolio over time, then biased results will occur in the estimates of the beta coefficients of the market model.

B.J.S., B.F. and F.M. used similar grouping procedures and they generated 10,12, and 20 portfolios respectively. B.J.S. estimated portfolio betas by utilizing the ex-post form of the r_F-model. B.F. estimated portfolio betas with the help of the market model. Lastly F.M. estimated each portfolio's beta by averaging the individual security betas in the portfolio; the individual security betas were estimated with the aid of the market model.

The cross-section methodology is not the only methodology adopted to test the C.A.P.M.'s versions. B.J.S. presented an additional methodology called the time series methodology. With this methodology in mind the following procedure was adopted:

STEP 1 For each portfolio in the sample, the coefficients alpha and beta of a time series regression between portfolio realized returns and realized excess returns on the market index were calculated.

STEP 2 The vector with entries the alpha coefficients was compared with the zero vector.

To investigate empirically the second implication of the C.A.P.M.'s versions a three-stage methodology was used :

STEP 1 For each portfolio in the sample the beta coefficient and the specific risk were estimated.

STEP 2 A portfolio's average realized return or porfolio's realized return was used as dependent variable and was cross-sectionally regressed against two independent variables: the estimated portfolio beta and the estimated portfolio specific risk.

STEP 3 The coefficients of the cross-sectional regressions were compared with the risk-free rate, the mean excess return on the market index and the number zero, respectively.

This methodology was adopted by Lintner, Miller and Scholes (M.S.) and F.M. Lintner and M.S. used securities in their tests, while F.M. used portfolios. Lintner and F.M. employed the market model to estimate security betas and security specific risks. M.S. employed, for the same purpose, the ex-post of the r_F-model. F.M. estimated each portfolio's beta by averaging the individual security betas in the portfolio and each portfolio's specific risk by averaging the individual security specific risks in the portfolio.

Finally, the third implication of the C.A.P.M.'s versions can be verified by examining the sign of the second coefficient of the cross-sectional regressions employed to investigate empirically the first and the second implications. The results of the C.A.P.M.'s versions tests are summarized in table 3.1 .

Table 3.1 Summary results of the capital asset pricing model's versions tests

STUDY OF	DECISION CONCERNING THE ACCEPTANCE OR REJECTION OF A LINEAR RELATIONSHIP BETWEEN AVERAGE RETURN AND BETA	DECISION CONCERNING THE ACCEPTANCE OR REJECTION OF THE IMPLICATION THAT BETA IS THE ONLY MEASURE OF RISK RELATED TO AVERAGE RETURN	DECISION CONCERNING THE ACCEPTANCE OR REJECTION OF A POSITIVE RELATIONSHIP BETWEEN AVERAGE RETURN AND BETA	DECISION CONCERNING THE ACCEPTANCE OR REJECTION OF THE r_F- MODEL	DECISION CONCERNING THE ACCEPTANCE OR REJECTION OF THE r_Z- MODEL
			TESTS BASED ON SECURITIES		
Lintner	accepted	rejected	accepted	rejected	–
Jacob	accepted	–	accepted	rejected	–
Miller and Scholes	accepted	rejected	accepted	rejected	–
			TESTS BASED ON PORTFOLIOS		
Black Jensen and Scholes	accepted	–	accepted	rejected	accepted
Blume and Friend	accepted	accepted	accepted	rejected	–
Fama and MacBeth	accepted	accepted	accepted	rejected	–

3.3 Roll's Criticisms

The conclusions for accepting or rejecting the versions of the C.A.P.M. were carried out until Roll's criticisms appeared in the literature.

Roll (1977, 1978) pointed out that :

(1) Tests of the C.A.P.M. are impossible and invalid.

(2) There are difficulties in testing the efficiency of the market proxy.

(3) When a market proxy is used as a benchmark, superior performance will be undetected.

Roll's main conclusion can be summarized as follows :

Assume a given number of N risky securities. A portfolio, call it M1, is mean-standard deviation boundary (i.e.it has a minimum standard deviation at each level of mean return) if, and only if, there exists an exact linear relationship represented by the following equation :

$$R = r_{ZM} i + (r_{M1} - r_{ZM}) \frac{VX_{M1}}{\sigma^2_{M1}} \qquad (3.9)$$

where

R = the (N x 1) column vector of mean returns.

V = the (N x N) covariance matrix of returns.

i = the (N x 1) unit vector.

X_{M1} = a (N x 1) decision column vector of investment proportions defining the portfolio M1.

r_{M1} = the (scalar) mean return of portfolio M1.

σ^2_{M1} = the (scalar) variance of portfolio M1.

r_{ZM1} = the (scalar) mean return on a boundary portfolio whose return is uncorrelated with the return on M1.

Roll's conclusion holds in the population of securities in the market as well as in any sample drawn from this population. The two mathematical assumptions on which this conclusion is based are :

(1) The covariance matrix of returns on the securities is non-singular.

(2) The mean return vector contains at least two distinct entries.

These assumptions are not additional assumptions. The derivation of the C.A.P.M. relies on the non-singularity of the covariance matrix of security returns. Furthermore, there are two securities in the whole population of securities in the market with different returns. However, these assumptions are made when one wishes to prove Roll's main conclusion for a given sample of securities.

The consequences of Roll's conclusion, are very important. In fact it was stated previously that the market portfolio's efficiency and only the market portfolio's efficiency will imply the C.A.P.M. In addition the converse proposition is true ; that is the validity of the C.A.P.M. implies the market portfolio's efficiency. Then, in view of this result, it may be concluded that the ex-ante efficiency of the market portfolio and the validity of the C.A.P.M. are joint hypotheses. Consequently the only way to test the C.A.P.M. directly is to test the following implication :

(1) The market portfolio is mean-standard deviation efficient.

But the true market portfolio contains all the market risky securities in proportion to their relative value in the market place. This implies that one can test the C.A.P.M. directly if given all the securities that comprise the market portfolio and the equilibrium proportions of each security in the market portfolio. Since it is not possible to identify all the market securities and hence to compute their weights in the market portfolio, it is impossible to identify the market portfolio itself. Therefore it is impossible to validate a model whose testability relies upon the identification and use of an unobservable portfolio.

All the above mentioned studies are based on a sample of risky securities with observed returns over a sample period and a proxy which was supposed to represent the market portfolio. But by using a proxy nothing can be deduced about the C.A.P.M. Furthermore, since the market portfolio is unobservable no inference can be made for a given proxy, that it is a good or bad approximation of the market portfolio. In this case two conclusions can only be drawn. Firstly, if the proxy is efficient during the sample period, it can be concluded the validity of an exact sample risk-return relationship during this period, whether or not the market portfolio is mean-standard deviation efficient. Secondly, if it is found that a sample risk-return linearity holds exactly using a market proxy, it can be inferred that during the sample period this proxy is a member of the sample mean-standard deviation efficient frontier. But this does not mean that the market portfolio is also a member of the mean-standard deviation efficient frontier.

Unfortunately, all the studies which tried to test the C.A.P.M. paid little attention to this crucial point. It would, therefore, be extremely misleading to draw any conclusions about the validity of the C.A.P.M. itself. In fact, these studies provide nothing more but tests of the mean-standard deviation efficiency of the market proxy.

Consequently it can be deduced that the problem of the market portfolio identification implies that both versions of the C.A.P.M. are only theoretical models.

The same conclusion applies for the Sharpe multi-beta model, since it also takes into consideration the market portfolio.

Next attention is given to the difficulties which may appear when the efficiency of the market proxy is tested. Indeed :

(I) By testing directly the mean-standard deviation efficency of the market proxy, there are computational problems. In this case one has to compute the efficient frontier and to infer whether the market proxy lies on the frontier. Such a procedure, however, requires the inverse of the covariance metrix of all securities in the sample.

(II) By testing the efficiency of the market proxy with the aid of the return-beta linear relationship it can be alleviated the problem concerning the computation of the covariance matrix's inverse. However, some other statistical problems arise. Namely there exists a number of inefficient portfolios for which the usual statistical tests of the equation $R-r_F i = A_s + (r_{M1} - r_F)\ VX_{M1}/\sigma^2_{M1}$ will give $A_s = \underline{0}$, where s is a portfolio formed from securities contained in the sample and $\underline{0}$ is a zero vector.

This is true since the vector A_s contains some positive and some negative entries and thus its mean would be equal to zero. Consequently the efficiency of the market proxy may be accepted when the market proxy is not efficient.

Finally, with regard to the portfolio performance, Roll concluded :

(1) If the selected proxy is ex-post efficient, then all individual securities and portfolios (efficient or not) would lie on a straight line derived from the efficiency of the proxy. Hence all the individual measures of performance will be zero. Therefore, in this case, it will be impossible to find a security or portfolio (efficient or not) with superior or inferior performance.

(2) If the selected proxy is not ex-post efficient, the efficient set mathematics implies that there is not an exact linear risk-return relationship. Thus there exist non-zero individual measures of performance. But another ex-post inefficient proxy will also give non-zero individual measures of performance. In such a case there is not a method to find out which proxy has to be used.

To distinguish between the C.A.P.M., which applies to the whole population of securities in the market and the risk-return exact linear relationship, which applies to a sample of securities the latter is termed Sample Risk-Return Exact Linear Relationship (S.R.R.E.L.R.).

3.4 Capital Asset Pricing Model versus Sample Risk-Return Exact Linear Relationship

If one compares the C.A.P.M. and the S.R.R.E.L.R. the following is observed :

(1) The C.A.P.M. requires all the population of the risky securities in the market to be included in the market portfolio. The S.R.R.E.L.R. considers a sample of the population of the risky securities in the market to be included in the market proxy.

(2) The C.A.P.M. arises from the expected return-standard deviation efficiency of the market portfolio.
The S.R.R.E.L.R. is a consequence of the expected return-

standard deviation efficiency of the market proxy.

(3) The C.A.P.M. can be viewed as a model which applies equally to all investors.
The S.R.R.E.L.R. can be considered as a model which applies to an individual investor.

(4) The C.A.P.M. is an ex-ante equilibrium model.
The S.R.R.E.L.R. is an ex-ante mathematical model.

(5) The C.A.P.M. is a unique model in the sense that it holds for the market portfolio and only the market portfolio.
The S.R.R.E.L.R. holds for an infinite number of expected return-standard deviation efficient market proxies.

(6) The market portfolio is a theoretical portfolio and it cannot be identified . Hence the C.A.P.M. is not testable.
The market proxy can be identified and thus the S.R.R.E.L.R. may be tested.

3.5 Some Other Criticisms of the Capital Asset Pricing Model's Tests

In all the empirical studies mentioned in Section 3.2 samples were used and time series regressions were run to estimate the beta coefficients. The estimating method employed was the Ordinary Least Squares (O.L.S.) method.
A major justification for using the O.L.S. method to estimate the linear regression is the following :

The least square estimates of a linear regression are Best Linear Unbiased Estimators (B.L.U.E.s) of the true population parameters.

The properties of B.L.U.E.s were important for the empirical investigations of the C.A.P.M. (S.R.R.E.L.R.) of section 3.2, because such investigations were based upon the estimation of linear regressions. If the properties of B.L.U.E.s were violated, biased sample estimators would be derived and so standard formulae and tests for statistical inference would be invalid.

The estimation of a linear regression by O.L.S. is subject to a set of assumptions, which in turn implies that the O.L.S. estimators are B.L.U.E.s. These assumptions are :

(1) <u>The Normal Distribution Assumption</u> : The random disturbance term is normally distributed.

(2) <u>The Zero Expected Value Assumption</u> : The expected value of the disturbance term is equal to zero.

(3) <u>The Homoscedasticity Assumption</u> : The variance of the disturbance term is intertemporally stationary.

(4) <u>The Non-Autocorrelation Assumption</u> : The value which the disturbance term takes in one period is independent of its value in any other period.

(5) <u>The Assumption of Independence</u> : The disturbance term is independent of the explanatory variable.

Unfortunately no attention was given to these assumptions by any study mentioned in section 3.2. Only Miller and Scholes (1972) examined the homoscedasticity assumption, but they did not find evidence to support it. However, there is evidence in the literature which rejects the simultaneous validity of these assumptions. Brown (1977) used the data of Fama and MacBeth (1973) for the period 1961 to 1968. His empirical evidence showed heteroscedasticity for securities at the one percent level of significance and for portfolios at the five percent level of significance.

Bey and Pinches's (1980) results also supported the existence of heteroscedasticity when individual securities were used. In addition they arrived at the same conclusion when they employed the single-index model using portfolios instead of individual securities.

Belkaoui's (1977) findings also indicated evidence of heteroscedasticity.

Schwartz and Whitcomb (1977) examined empirically the non-autocorrelation assumption. Their reporting evidence showed autocorrelation between the residuals.

Theobald (1980) investigated empirically the assumptions of non-autocorrelation and homoscedasticity using U.K. data. In view of his findings he deduced that during his sample period both assumptions were violated.

On the other hand the empirical studies of the C.A.P.M. (S.R.R.E.L.R.) utilized cross-sectional regressions. Unfortunately, the cross-sectional regression's assumptions under which the B.L.E.U.s' properties of the estimators are insured have been left untested. For example Fama and MacBeth (1973) in their footnote 10, p.627, stated :

> "If one makes the Gauss-Markow assumptions that the underlying disturbances $\tilde{\eta}_{pt}$ of (11) have zero means, are uncorrelated across p, and have the same variance for all p..." .[1]

In another paper Fama and MacBeth (1974) in their footnote 20, p.867, stated :

> "There is, however, ample evidence in (7) that the assumptions of the Gauss-Markow Theorem are not by the $\tilde{\eta}_{it}$ of (8). In particular, $\sigma^2(\eta_i)$ is an increasing function of b_i, and there is a slight amount of cross-correlation among the $\tilde{\eta}_i$. This implies that to obtain $R_{Zt} = \hat{\gamma}_{ot}$ the coefficients of (8) must be estimated by generalized least squares" .[2]

Recently Chen (1981) demonstrated that the effect of portfolio diversification cannot be achieved if a portfolio is comprised of securities among which a large number have heteroscedastic residual risk over time.

In another study Chen (1980) concluded that the O.L.S. method is not an appropriate method for use in estimating beta, since the disturbance terms of the ex-post form of the C.A.P.M. did not satisfy the assumption of homoscedasticity.

1 Their equation (11) was a second pass cross-sectional regrssion used to test the C.A.P.M.

2 Their equation (7) was the C.A.P.M. ,while treir equation (8) was a second pass cross-sectional regression used to test the C.A.P.M.

He proposed the use of the generalized least squares method to estimate the beta coefficients and the disturbance terms. By employing the method of the generalized least squares he re-examined the Lintner's (in Douglas (1968), p.36) results and he deduced that average security returns were not significantly related to the specific risk.

Lastly the one-period C.A.P.M. (S.R.R.E.L.R.) derived without invoking the assumption that the joint distribution of security returns is intertemporally stationary. But this assumption is required when one uses time series data to test the model, because it is a necessary condition to derive the ex-post form of the C.A.P.M. (S.R.R.E.L.R.) and to estimate the security beta coefficients and the security specific variances by the O.L.S. method. (More will be said about such an assumption in Chapter 9).

Although the intertemporal stationary assumption is very important for the empirical examination of the C.A.P.M. (S.R.R.E.L.R.), it has been left untested. There are only some indirect tests in the literature concerning the empirical verification of the security variance.

Blume (1975) and King (1966) found that the variance of security returns on the New York Stock Exchange decreased from the pre-war to the post-war period.

Also Blattberg and Conedes (1974) reported evidence indicating that the individual security variance is not intertemporally stationary.

However, when one examines empirically one-period asset pricing models he is concerned with the joint probability distribution of security returns and not only with the intertemporal stationarity of the individual security variances. Consequently before performing a test of the C.A.P.M. (S.R.R.E.L.R.) it is necessary to verify empirically the assumption concerning the intertemporal stationarity of the security mean returns and the covariance matrix of security returns.

3.6 Criticisms of Some Assumptions of the Capital Asset Pricing Model

Below are summarized some criticisms about the set of assumptions of the C.A.P.M.

(1) The C.A.P.M. puts restrictions on the type of the joint probability distribtion of returns on securities (e.g. multivariate normal) or on the type of the investor's utility function of portfolio returns (e.g. quadratic).

The assumption of the multivariate normal distribution of security returns has been questioned in the literature. A number of studies have examined series of daily price changes. The results indicated that such price changes are not normally distributed, see Fama (1965), Lintner (1972), Mandelbrot (1963), Officer (1972), Press (1964), Rosenberg (1973) and Teichmoeller (1971).

The assumption of the quadratic utility function has also been questioned, see Mossin (1966) and Pratt (1964).

(2). The C.A.P.M. assumes that all securities in the market are marketable. However this assumption cannot be justified in practice because there are securities which cannot be traded in the market. Human capital is an example of such a security. You may rent your skills in return for wages, but it is forbidden by law to sell yourself or buy anyone else.
Thus the human capital of an investor constitutes one of the risky securities contained in his portfolio.
Since the human capital of an investor cannot be owned by another investor, one may conclude that investors hold different portfolios of risky securities. This is contrary to the C.A.P.M. implication which states that all the investors in the market hold the same portfolio of risky securities [1].

(3) The C.A.P.M. uses at the most two factors that affect security returns. There is, however, empirical evidence which shows the existence of more factors. (See King (1966), Farrel (1974), Aber (1976), Roll and Ross (1980) and Hughes (1982)).

3.7 Conclusions

Empirical evidence indicates that there are cases where a multi-index model outperforms a single-index model and cases where a multi-index model is outperformed by a single-index model. There are not however,unambiguous answers regarding

---

1. Mayers D. (1972, 1973) derived the C.A.P.M. by relaxing the assumption that all securities in the market are marketable.

the performance of the multi-index models relative to the single-index models.

On the other hand both versions of the C.A.P.M. are only theoretical models and they cannot be tested, because the market portfolio cannot be identified. Hence all the tests performed were simply tests of the mean-standard deviation efficiency of the chosen market proxy.

Lastly, the S.R.R.E.L.R. is not an appropriate tool for for assessing investment performance.

The C.A.P.M. is vulnerable to criticisms, particularly that concerning its testability. So attention has to be given to other models that relate linearly expected return and risk. The S.R.R.E.L.R. has the advantage over the C.A.P.M. that it avoids the problem of the market portfolio's identification ; hence it may be tested. But it also has some disadvantages as against the C.A.P.M. It is a purely mathematical relationship which cannot be considered as an equilibrium relationship. It is valid for any minimum standard deviation portfolio in a sample, provided that the covariance matrix of security returns is non-singular and there exist at least two securities with different expected returns. Therefore it may be deduced that the S.R.R.E.L.R. cannot be regarded as a good substitute for the C.A.P.M.

What is needed is a multi-factor model which :

(1) Overcomes the criticisms of the C.A.P.M. and S.R.R.E.L.R.

(2) Relates linearly security expected returns and risks as measured by the sensitivities of the security returns to the fluctuations in common factors.

A model which fulfils these required conditions is the arbitrage pricing model, developed by Ross (1976, 1977). The rest of this research is devoted to the arbitrage pricing model.

CHAPTER 4

THE ARBITRAGE PRICING THEORY

According to Roll (1977) the C.A.P.M. cannot be tested because the market portfolio cannot be identified. Another alternative to the C.A.P.M., which has recently attracted a great deal of interest, is the Arbitrage Pricing Model (A.P.M.), presented by Ross (1976, 1977).

Ross' model is an approach to general equilibrium theory that overcomes most criticisms of the C.A.P.M. and especially the problem of the market portfolio's identification.

This chapter discusses in detail the Arbitrage Pricing Theory (A.P.T.). It also proves that the A.P.M. holds under a weaker "no arbitrage" condition than that of Ross.

4.1 Notation

Suppose there are in an economy N risky securities, where $N \in \{ n : n \text{ is a large integer} \}$. In this chapter the following notation is adopted, unless otherwise stated or implied :

R : the (Nx1) column vector with components the actual returns on the N securities at period t.

R_E: the (Nx1) column vector with components the single period expected returns on the N risky securities.

δ_K: the value of an unknown common factor affecting the security returns during the period t, $K = 1,2,\ldots,k$.

δ: the (Kx1) column vector whose components are δ_K, $K = 1,2,\ldots,k$.

${}^K b$: the (Nx1) column vector with components the security response coefficients to changes in a common factor.

B : the (NxK) matrix whose columns are the (Nx1) vectors ${}^K b$.

b_i: the (Nx1) column vector with components the security i's response coefficients to changes in the common factors $1,2,\ldots,k$.

e : the (Nx1) column vector with components the security specific disturbances.

i : the (Nx1) unit vector.

I : the (NxN) identity matrix.

$\underline{0}$: the (Nx1) zero vector.

$\emptyset$: the (NxN) zero matrix.

E : the expectation operator.

4.2 Assumptions Underlying the K-Factor Arbitrage Pricing Model

The K-Factor Arbitrage Pricing Model relies upon the following assumptions :

(A) Assumptions about investors

(1) The Generating Process of Security Returns Assumption : Investors believe that the (Nx1) column vector of security's ex-post return in period t is described by the following equation :

$$\tilde{R} = R_E + B\tilde{\delta} + \tilde{e} \qquad (4.1)$$

where the tilde " $\sim$ " indicates that the entries of each vector are random variables.

It is also assumed that :

(i) The single period expected values on the factors $\tilde{\delta}_1, \tilde{\delta}_2, \ldots, \tilde{\delta}_K$ are zero. That is $E(\tilde{\delta}) = \underline{0}$.

(ii) For each security i, the expected value of the distribution of $\tilde{e}_i$ conditional on δ_K, for all K, is zero. That is $E(\tilde{e}_i / \tilde{\delta}_K) = 0$, for all K .

(iii) The entries of $\tilde{e}$ are commonly distributed. The disturbance of a security is independent of any other security's disturbance and each disturbance has finite variance, say $\sigma^2_{e_i}$, i = 1,2,...,N. That is $E(\tilde{e}\tilde{e}') = U$, where U is an (NxN) diagonal matrix with the variances $\sigma^2_{e_i}$ along the diagonal.[1]

1(a) In the remaining part of this study transposition of vectors or matrices will be denoted " ' ".

(b) In contrast to the theory behind the C.A.P.M. the theory behind the A.P.M. does not require the use of the variance. Consequently for the theoretical validity of the A.P.M. the assumption concerning the independence of the random disturbances and $\tilde{\delta}_1, \tilde{\delta}_2, \ldots, \tilde{\delta}_K$ is not necessary.

(2) The Expected Return Preference Assumption : Investors prefer higher expected return to lower.

(3) The Risk-Aversion Assumption : Investors are risk-averse.

(4) The One-Period Assumption : Investors are assumed to have a one-period investment horizon.

(B) Assumptions about the Market

(5) The Short-Selling Assumption : Short-selling is unrestricted.

(6) The Perfect Capital Market Assumption : The capital market is perfect in the sense discussed in Chapter 2.

(7) The Taxless Assumption : There are no taxes.

(8) The "no arbitrage" Condition Assumption : Investors cannot earn in the market a positive return with no risk and without using their own wealth.

(C) Mathematical Assumptions

(9) The Difference of Expected Returns Assumption : There are at least two securities among the N securities with different expected returns. Stated in another way the rank of the (Nx2) matrix (R i) is equal 2.

(10) The Number of Factors Assumption : There are K distinct and no perfectly correlated factors which influence security returns. Viewed differently

the rank of the Nx(K+1) matrix (B i) is K + 1[1] .

4.3 Implications of the Assumptions of the K-Factor Arbitrage Pricing Model

From equation (4.1) one observes that in the economy the t^{th} period ex-post return on any security, say i, is represented by the following equation :

$$\tilde{R}_{it} = r_i + b_i' \tilde{\delta} + \tilde{e}_{it} \tag{4.2}$$

Equation (4.2) shows that R_{it} is described linearly in terms of K-random variables common to all securities, called <u>factors</u>, and a random variable specific to security i, called the <u>idiosyncratic risk</u> of i.

Since R_{it} is related to random variables, it is a random variable itself. The random variables $\tilde{\delta}_{1t}, \tilde{\delta}_{2t}, \ldots \tilde{\delta}_{Kt}$ and $\tilde{e}_{it}$ can be thought of as governed by a probability distribution. In A.P.T. there is no specific assumption about the type of the probability distributions of $\tilde{\delta}_{1t}, \tilde{\delta}_{2t}, \ldots, \tilde{\delta}_{Kt}$ and $\tilde{e}_{it}$. It is only assumed that such distributions have well defined variances. Therefore except for equation (4.2) there is no other restriction on the type of the probability distribution of $\tilde{R}_{it}$. Furthermore, the distribution of $\tilde{R}_{it}$ has a well defined variance, since the distributions of $\tilde{\delta}_{1t}, \tilde{\delta}_{2t}, \ldots, \tilde{\delta}_{Kt}$ and $\tilde{e}_{it}$ have well defined variances.

---

1. Although the assumptions (9) and (10) were not mentioned by Ross it will be shown (Appendix A) that they are necessary.

In view of equation (4.2) the variability of future returns on i (i.e. its riskiness) is dependent on the variabilities of $\tilde{\delta}_{1t}, \tilde{\delta}_{2t}, \ldots, \tilde{\delta}_{Kt}$ and $\tilde{e}_{it}$. In other words, from equation (4.2) it is clear that the extent to which the security i's return fluctuates is dependent on the extent to which $\tilde{\delta}_{1t}, \tilde{\delta}_{2t}, \ldots, \tilde{\delta}_{Kt}$ and $\tilde{e}_{it}$ fluctuate. Consequently, the risk of the individual security i can be separated into two components : a systematic component due to the movements in the common factors, called <u>systematic or factor risk</u> and the unsystematic component due to the movements in e's (the idiosyncratic risk)[1] .

In equation (4.2) a typical element b_{iK} of the vector b_i measures the sensitivity of the security i's return to fluctuations in the common factor $\tilde{\delta}_K$. b_{iK} is called <u>the security i's response coefficient to fluctuations in the common factor $\tilde{\delta}_K$, or the K-factor beta coefficient for the i^{th} security</u>.

The restriction $K \geqslant 1$ requires that there exists at least one common factor that accounts for the correlation between the N risky securities. There are also two justifications behind the assumption $K < N$. These are :
(i) To transform the A.P.M. into a testable relationship. The A.P.M. can be tested by using factor analysis techniques, which in turn assume $K < N$.

---

1. Since by assumption (liii) the disturbances are independent of each other one may infer that $\tilde{e}_{it}$ generates the idiosyneratic risk of the security i.

(ii) To ensure that it is dealt with a satisfactory model. The A.P.M. would be characterized as a satisfactory model if there exist few common factors having influence on security returns and these factors'variability explains a large portion of the total variabilities on securities.

All the factors $\delta_{1t}, \delta_{2t}, \ldots, \delta_{Kt}$ do not have to be actual security portfolios. For example some of the factors might be:the inflation rate, the gross national product, the unemployment rate, changes in interest rates, etc..

Equation (4.2) describes a stochastic model that relates ex-post returns to ex-ante returns on the i^{th} security. Such an equation is called a stochastic (or random) security returns generating model .

On the other hand equation (4.2) and the assumptions (1i) and (1ii) imply that, on average, the expected return on the security i is equal to its realized return. Thus, according to equation (4.2) the return on a security is a fair gain. In effect there is no way to use the ex-post return on a security i available at a point in time t to earn an expected return greater than r_i . Hence large arbitraging profits cannot be obtained, which is consistent with the assumption (8).

The assumption that the disturbance of a security is independent of any other security's disturbance implies that there are no factors beyond $\delta_{1t}, \delta_{2t}, \ldots, \delta_{Kt}$ that account for the correlation between the returns on any two securities.

Next an immediate implication of assumption (6) is that at any point of time one price of the individual security i is ruled in the market. This is perfectly consistent with equilibrium situations of the market.

Moreover, the perfect capital market assumption is consistent with the "no arbitrage" condition assumption. Indeed if a security has two different prices in the market at a point of time t, then investors could purchase the security at the lower price and sell the same security at the higher price. This means that at the point of time t there exist arbitrage conditions in the market. The previously mentioned discussion also shows that the existence of the " no arbitrage" condition in the market is also consistent with market equilibrium situations.

Next the security return generating model is a pure mathematical relationship on which relies the A.P.M. . In this case the assumption (6), that is investors cannot earn in the market a positive return with no risk and without using their own wealth, gives to the A.P.M. an economic content. This implies that the A.P.M. is not a pure mathematical relationship and thus it has an empirical content.

Lastly since the capital market is perfect there are no transaction costs. Thus if there exists a riskless secuity then the riskless rate of borrowing and the riskless rate of lending are equal. This can be explained as follows : suppose an investor borrows an amount from a borrower and there is a broker who charges a fee from both the borrower and the lender. Then the borrower does not realize the full amount of the loan and the lender has to pay the broker a fee in addition to the face value of the loan. Hence in this case the riskless borrowing rate is greater than the riskless lending rate[1] .

If X_{pl} is a (Nxl) column vector of investment proportions, defining a portfolio pl then in view of equation (4.1) one takes :

$$\tilde{R}_{plt} = X_{pl}\tilde{R} = r_{pl} + B_{pl}\tilde{\delta} + \tilde{e}_{plt} \tag{4.3}$$

where

R_{plt} = the return on portfolio pl in period t.

r_{pl} = the one period expected return on pl, with

$$r_{pl} = X'_{pl}R_E \tag{4.4}$$

$$B_{pl} = X'_{pl}B \tag{4.5}$$

$$\tilde{e}_{plt} = X'_{pl}\tilde{e} \tag{4.6}$$

$$X'_{pl}i = 1 \tag{4.7}$$

---

1. The implications of the remaining assumptions will be explained further in this chapter and in Appendix A.

Equation (4.3) indicates again that R_{p1t} is a random variable. Here also the essential restriction on the probability distribution of $\tilde{R}_{p1t}$ is equation (4.3). The distributions of $\tilde{\delta}_{1t}, \tilde{\delta}_{2t}, \ldots, \tilde{\delta}_{Kt}$ and $\tilde{e}_{it}$ have well defined variances and so the distribution of $\tilde{R}_{p1t}$ has a well defined variance. Equation (4.3) is also a fair game and thus it is consistent with the assumption (8). The risk of the portfolio p1 can be broken into two components : a common factor component generating by the common factors and a specific factor component generated by the $\tilde{e}_{p1t}$.

The development of the theory of the A.P.M. is started by setting down the following definitions :

DEFINITION 4.1 An arbitrage portfolio, call it a1, is defined or characterized by a non-zero (Nx1) column vector y_{a1} with

$$y'_{a1} i = 0$$

Since short-selling is permitted (see assumption (5)) the investors can sell some of their securities and use the proceeds to buy others. Hence one can always define a (Nx1) vector y_{a1} satisfying the last equation. For example, for an equally weighted arbitrage portfolio generated by N securities, where N is an even natural number, one can write :

$$y_{a1} = \begin{bmatrix} \frac{1}{N} \\ \vdots \\ \frac{1}{N} \\ -\frac{1}{N} \\ \vdots \\ -\frac{1}{N} \end{bmatrix} \tag{4.8}$$

where the number of the positive weights is equal to the number of the negative weights.

DEFINITION 4.2 An arbitrage portfolio comprised of N securities, whose returns are generated by the K-factor model is called a zero-systematic risk portfolio if :

$$y'_{a1} B = 0$$

it is always possible to choose a non-zero vector y_{a1}, such that :

$$y'_{a1} i = 0$$

$$\text{and} \quad y'_{a1} B = 0$$

This is true since :

(i) $K < N$

(ii) The rank of the $N x (K+1)$ matrix $(B \ i)$ is $K + 1$.

It also becomes clear that if either $K = N$ or $K = N - 1$ it cannot be found an $(Nx1)$ non-zero vector y_{a1} such that $y'_{a1} i=0$ and $y'_{a1} B=0$.

DEFINITION 4.3 A portfolio whose return is given by equation (4.3) can be classified as a well diversified portfolio if :

(i) it contains a large number of securities.

(ii) its security disturbances are mutually independent.

That is $E(e_{it} e_{jt}) = 0$ for all pairs of securities $i = 1,2,\dots,N$ and $j = 1,2,\dots,N$, but $i \neq j$.

DEFINITION 4.4 An arbitrage portfolio containing N securities whose return are generated by the K-factor model is called riskless if :

(i) it is a zero-systematic risk portfolio.

(ii) its idiosyncratic variance approaches to zero.

DEFINITION 4.5 In an economy there is a "no arbitrage" condition if each riskless arbitrage portfolio has a zero expected return.[1]

With the aid of assumptions (1) - (10) the following equation may be derived :

$$r_i \approx r_Z + b_{i1}(r_{L1}-r_Z)+b_{i2}(r_{L2}-r_Z)+\ldots+b_{iK}(r_{LK}-r_Z) \qquad (4.9)$$

where

$i = 1,2,\ldots,N$.

r_{LK} = the expected return on a portfolio with systematic risk equal to one on factor K and no risk on the remaining factors K-1 .

r_Z = the expected return on a portfolio that is orthogonal with each portfolio LK, for each K.

b_{iK} = the K-factor beta coefficient for the i^{th} security.

---

1. Huberman (1981) pointed out that Ross (1976) did not give an explicit definition of arbitrage. Huberman defined arbitrage as the existence of a subsequence of arbitrage portfolios which have an infinite return and zero variance as the number of securities in the portfolios approaches to infinity.

Equation (4.9) shows that the expected return on the security i is approximately a linear combination of K-factor beta coefficients. Equation (4.1) indicates that the risk of the security i can be divided into systematic and idiosyncratic risk. Then the A.P.M. proves that systematic risk is the only important ingredient in determining returns and that the idiosyncratic risk plays no important role. This is true since the idiosyncratic risk can be diversified and hence the market will not offer any compensation to the investor for bearing this type of risk.

Let it be assumed that the total risk of a security is measured by the variance of returns. Then it can be inferred that the variance of returns does not affect the security's expected return. Expected security returns can only be affected by that risk which cannot be eliminated by portfolio diversification.

A factor K which has a risk that cannot be diversified away will earn a risky return in the market. Hence, according to equation (4.1), the security returns are dependent upon this risky return. If the factor K is "priced" in the market the equation of the A.P.M. will contain a non-zero element called the risk premium on factor K.[1] That is a risk premium on factor K is equal to the difference between the expected return on K and the expected return on the orthogonal portfolio of K multiplied

1. A non-priced factor may be necessary to explain a security's stochastic return, but not necessary to explain the security's expected return.

by the K-factor beta coefficient. The A.P.M. assumes that investors are risk-averse. Therefore a risk premium on factor K is the additional required expected return by the risk-averse investor, to compensate him for undertaking an additional risk due to the riskiness of the factor K. The A.P.M. also implies that in a market of risk-averse investors, higher risk should always be associated with higher expected returns.

Equation (4.9) defines asymptotically a hyperplane in a (K+1)-dimensional space, called the K-arbitrage pricing hyperplane.

Equation (4.9) is equivalent to the following equation :

$$r_i \approx (1-b_{i1}-b_{i2}-\dots-b_{iK})r_Z+b_{i1}r_{L1}+b_{i2}r_{L2}+\dots+b_{iK}r_{LK} \quad (4.10)$$

In words equation (4.10) states that the expected return on the security i can be approximately expressed as a linear combination of the expected returns on K + 1 portfolios.

Next let pl be a portfolio defined by a (Nxl) column vector X_{pl}. Then equation (4.9) produces :

$$r_{pl} \approx r_Z+b_{p1}(r_{L1}-r_Z)+b_{p2}(r_{L2}-r_Z)+\dots+b_{pK}(r_{LK}-r_Z) \quad (4.11)$$

where

r_{pl} = the one-period expected return on pl, with

$$r_{p1} = X'_{p1} R_E$$

$$b_{pk} = X'_{p1} \overset{k}{b}$$

$$k = 1,2,\ldots,K \ .$$

Equation (4.11) shows that the expected return on the portfolio p1 can be approximately represented by a linear combination of the factor beta coefficients $b_{p1}, b_{p2}, \ldots, b_{pK}$.

Lastly a special case of equation (4.9) is the case where there exists only a single common factor affecting the security returns. In such a case equation (4.9) becomes :

$$r_i \approx r_Z + b_{i1} (r_{L1} - r_Z) \qquad (4.12)$$

where

$i = 1,2,\ldots,N$.

Furthermore if the whole population of securities in the market is taken into consideration then one can use the same procedure as above to prove :

$$r_i \approx r_Z + b_{iM} (r_M - r_Z) \qquad (4.13)$$

where

b_{iM} = the security i's response coefficient to fluctuations in the market portfolio.

r_M = the expected return in the market portfolio.

---

1. Since the A.P.M. relies on a return generating model having unobservable factors and since the market portfolio is unobservable it can be assumed a single return generating model with the market portfolio as a common factor, provided that the whole population of securities in the market is used.

Equation (4.13) shows that the expected return on the security i is approximately linear to the security i's response coefficient to fluctuation in the market portfolio. It is important to note that equation (4.13) is not the expected return standard deviation C.A.P.M., but it is the single factor arbitrage pricing model with common factor the unobservable market portfolio (more will be said in the next chapter).

4.4 A Weaker "no arbitrage" Condition and its Implications

Using the following "no arbitrage" condition :

(1) All portfolios that use a positive amount of wealth and have the same risk earn the same expected return.

A special case of this "no arbitrage" condition can be described as follows :

(2) All portfolios that use a positive amount of wealth and have zero risk earn the same expected return.

The "no arbitrage" condition (2) is weaker than the Ross' "no arbitrage" condition. This is true because Ross' "no arbitrage" condition can be applied when there exists only one arbitrage riskless portfolio, while the "no arbitrage" condition (2) cannot be applied in this case.
The "no arbitrage" condition (2) is called a weaker "no arbitrage" condition.

There is another difference between these two "no arbitrage" conditions. Ross' "no arbitrage" condition takes into account arbitrage portfolios, whereas the weaker "no arbitrage" condition does not take into consideration arbitrage portfolios.

The following conclusion can now be proceeded with :

Conclusion 4.4.1 The Ross' "no arbitrage" condition implies the weaker "no arbitrage" condition[1].

If the assumption of the weaker "no arbitrage" condition substitutes the assumption (8) of the A.P.M. it can be shown that the A.P.M. is still valid. If Ross' "no arbitrage" condition is fulfiled there is not any difference, since, according to conclusion 4.4.1, one can derive the A.P.M. by using Ross' "no arbitrage" condition or the weaker "no arbitrage" condition.

On the other hand the violation of the Ross' "no arbitrage" condition does not necessarily imply the theoretical invalidity of the A.P.M., since it can be derived by assuming the weaker "no arbitrage" condition.

A violation of Ross' "no arbitrage" condition is due to the absence of arbitrage portfolios, or the absence of (arbitrage) riskless portfolios. However, it cannot be concluded definitely that the weaker "no arbitrage" condition always can be used to solve the problem, since no riskless portfolios may exist.

Consequently the idea of introducing the weaker "no arbitrage" condition is to offer a possible alternative

1. The proof of this conclusion is given in Appendix A.

when Ross' "no arbitrage" condition is violated.[1]

4.5 Some Criticisms of the Arbitrage Pricing Model

The A.P.M. has received the following criticisms :

(1) The return generating model on which relies the A.P.M. contains a fixed number of distinct factors which cannot be observed.

(2) The A.P.M. is proved by applying the law of large numbers[2] which guarantees that a weighted average of security idiosyncratic risks will approach zero in the limit as the number of securities including in the arbitrage portfolio becomes very large. But such an approximation does not imply that every security's idiosyncratic risk approaches to zero. Considering an average over a large number of security idiosyncratic risks can produce some faulty predictions. For example, the A.P.M. could not be a good approximation for the expected returns on some securities if all others were exactly priced. This shows that the A.P.M. does not necessarily hold identically for all the N securities contained in an arbitrage portfolio.

(3) The A.P.M. provides a better approximation as the number of securities whose returns satisfy the linear generating model becomes very large. Therefore by testing

---

1. For the proof of the A.P.M. under the weaker "no arbitrage" condition see Appendix A.

2. See Appendix A.

the A.P.M. even in large capital markets (where there exists a large number of securities), it is not clear if the subset of securities which satisfy the return generating model is large enough to produce a reasonable approximation describing the security expected returns.

4.6 Conclusions

The A.P.M. was developed as an alternative model to the C.A.P.M. whose major disadvantage was the identification of the market portfolio. The A.P.M. rests upon a much simpler set of assumptions than the C.A.P.M. and it does not involve the use of the market portfolio.

The A.P.M. can also be derived by taking into account a weaker "no arbitrage" condition than that assumed by Ross.

CHAPTER 5

THE COMPARISON OF THE EXPECTED RETURN-STANDARD DEVIATION CAPITAL ASSET PRICING MODEL (SAMPLE RISK-RETURN EXACT LINEAR RELATIONSHIP) WITH THE ARBITRAGE PRICING MODEL

This chapter is devoted to the comparison between the C.A.P.M. (S.R.R.E.L.R.) and the A.P.M. Firstly it examines a "no arbitrage" condition behind the C.A.P.M. (S.R.R.E.L.R.) and it discusses if this "no arbitrage" condition is necessary for its deviation. It next describes the differences and similarities of the models before comparing the market model with the K-factor security return generating model of the A.P.M. There follows a discussion of the relationship between the C.A.P.M. (S.R.R.E.L.R.) and the A.P.M. and a statement with reference to misunderstandings about the relationship between the C.A.P.M. (S.R.R.E.L.R.) and the A.P.M. The chapter closes with a discussion about the empirical applications of the A.P.M.

5.1 "No arbitrage" Condition and Expected Return-Standard Deviation Portfolio Models

Let N be a number of risky securities, where $N \in \{ n$: n is a finite integer $\}$, and F_e be the set of expected return-standard deviation feasible portfolios. That is :

$$F_e = \{ X = (x_1, x_2, \ldots, x_N) : X'i = 1 \}$$

where

i = the (Nx1) unit vector.

By definition, the set of feasible portfolios does not contain arbitrage portfolios. Therefore it cannot be obtained a "no arbitrage" condition in the sense that riskless arbitrage portfolios have zero expected returns. It is possible, however, to observe a "no arbitrage" condition similar to the "no arbitrage" condition (1) or (2) presented in Chapter 4. That is :

(1a) All the expected return-standard deviation portfolios which use a positive amount of wealth and have the same risk in a Boundary Portfolio (B.P.), other than the Global Boundary Portfolio (G.B.P), measured relative to the risk of the B.P. earn the same expected return.[1]

A special case of the condition (1a) is the following :

(2a) All the expected return-standard deviation portfolios which use a positive amount of wealth and have zero risk in a B.P., other than the G.B.P., measured relative to the risk of the B.P. earn the same expected return.

The validity of the "no arbitrage" condition (1a) or (2a) can be proved by using the boundary portfolio set mathematics.[2] Therefore it may be deduced that behind the C.A.P.M. (S.R.R.E.L.R.) there is a "no arbitrage" condition which cannot be classified as an assumption, while this does not happen with the A.P.M.

1.(a) The definition of a B.P. is given in page 50 .
(b) Between the B.P.'s there exists one with a least standard deviation called the global boundary portfolio. The G.B.P. is excluded, since every portfolio(boundary or not) has the same risk in the G.B.P. measured relative to the risk of the G.B.P.

2. See Appendix B.

Furthermore this "no arbitrage" condition can be utilized to produce the C.A.P.M. (S.R.R.E.L.R.), provided that the market portfolio (market proxy) is expected return-standard deviation efficient. On the other hand it can be used to validate the expected return-standard deviation efficiency of the market portfolio (market proxy) provided that the C.A.P.M. (S.R.R.E.L.R.) holds.[1]

Since the C.A.P.M. (S.R.R.E.L.R.) can be derived by adopting for example the method presented by Roll (1977) or the method presented by Diacogiannis (1979) it can be deduced that such "no arbitrage" condition is not necessary for its derivation. This is in contrast to the A.P.M. which is based upon a "no arbitrage" condition.

5.2 Capital Asset Pricing Model versus Arbitrage Pricing Model

Comparing the C.A.P.M. and the A.P.M. it can be seen that:

(1) The C.A.P.M. is derived by considering the whole population of securities in the market.
The A.P.M. is derived by considering a large subset of the population of securities in the market.

(2) The C.A.P.M. rests primarily on the assumption that the distribution of security returns can be well approximated by a multivariate normal distribution or the utility function of portfolio returns is a quadratic approximation. Each

---

1. See Appendix B.

assumption implies decisions on the basis of means and variances of returns only. The theoretical validity of the C.A.P.M. does not rest either upon a generating model of security returns or upon Ross' "no arbitrage" condition.

Under the multivariate normality assumption of security returns a single factor generated model of security returns is implied (see equation (2.3)). But the theoretical form of the C.A.P.M. can be proved independently of such a model. Even if one could prove that the C.A.P.M. relies greatly upon such a model it proves nothing new. It has only proved that the C.A.P.M. is relying greatly upon the bivariate normality assumption between the return on each security and the return on the market portfolio.

On the other hand the quadratic utility assumption cannot produce any generating model of security returns or to ensure a condition similar to Ross' "no arbitrage" condition. Even if one assumes a multi-factor security return generating model the C.A.P.M. derivation can be achieved without taking into consideration such a model.

The A.P.M. does not require any restriction on the type of the joint distribution of security returns or on the type of utility function of portfolio returns . For the theoretical validity of the A.P.M. it is not necessary to assume that the joint probability of security returns is multivariate normal. It can be another distribution with well defined variance. Moreover for the A.P.M.'s theoretical validity the assumption of the quadratic approximation is not necessary. Another type of utility function of portfolio returns can be assumed provided that it is concave.

The only assumption of the A.P.M. about the joint probability distribution of security returns is the stochastic return generating model.

(3) The theory behind the C.A.P.M. requires the homogeneous anticipation assumption - all investors agree about both the expected return vector and the covariance matrix of security returns. Therefore the C.A.P.M. can be viewed as a model which applies equally to all investors.
The theory behind the A.P.M. does not require that investors have homogeneous anticipations. Namely investors may have the same ex-ante expectations, but still believe in different return generating models. Hence the A.P.M. can be considered as a model that applies to an individual investor.

(4) Both are static (one-period) models.

(5) The C.A.P.M. assumes that all securities in the market are are marketable.
The A.P.M. does not need such an assumption, because it can always be chosen a subset of the whole population of securities which contains only marketable securities.

(6) It can be proved that behind the C.A.P.M. there exists a "no arbitrage" condition (see Appendix B). The theoretical validity of the C.A.P.M., however, can be proved independently of this "no arbitrage" condition.
The A.P.M.'s basic set of assumptions contains the "no arbitrage" condition assumption. The theoretical validity of the A.P.M. relies upon such an assumption.

(7) The C.A.P.M. relies on the assumption of the market wide general equilibrium.
Since the A.P.M. is based on the assumption of the "no arbitrage" condition it does not need a market wide general equilibrium.

(8) The C.A.P.M. is an <u>exact</u> linear pricing ex-ante relationship.
The A.P.M. holds for large subsets of securities if there exists a well diversified arbitrage portfolio. In this case the law of large numbers can be applied, so that the idiosynciatic risk of the arbitrage portfolio approaches to zero. In consequence the arbitrage portfolio's expected return approaches to zero and hence the A.P.M. is an <u>approximate</u> linear pricing ex-ante relationship.

(9) The C.A.P.M. is an exact linear ex-ante relationship. Thus it holds identically for each security including in the market portfolio.
However, as it was explained in section 4.5 the A.P.M. does not necessarily hold identically for all the N securities contained in an arbitrage portfolio

(10) Both models have some common assumptions. These are the risk-aversion assumption, the short-selling assumption, the perfect market assumption and the taxless assumption.

(11) The C.A.P.M. uses only the market portfolio and it arises from the expected return-standard deviation efficiency of the market portfolio. It is a linear model relating

the expected return of a security or portfolio (efficient or not) to its relative risk in the market portfolio. So it has only one risk premium. Graphically in the plane (expected return-beta) the C.A.P.M. is a straight line. The A.P.M. uses more than one factor, none of which needs to be the market portfolio. In addition it does not arise from the expected return-standard deviation efficiency of the factors. It is a linear model, relating the expected return of a security (portfolio) to measures of the sensitivities of the return on the security (portfolio) to variations in the factors. Thus it has more than one risk premium. The A.P.M. defines asymptotically in a (K + 1)-dimensional space a hyperplane.

(12) The C.A.P.M. specifies the market portfolio's return in the risk premium. The A.P.M. does not specify the underlying factors in the risk premiums.

(13) In the C.A.P.M. the coefficient b_i measures the sensitivity of the security i's return to fluctuations in the market portfolio and it can be expressed as σ_{iM}/σ_M^2, where σ_{iM} is the covariance between the return on the security and the return on the market portfolio and σ_M^2 is the variance of the return on the market portfolio.
In A.P.M. $b_{i1}, b_{i2}, \ldots, b_{iK}$ are coefficient measuring the sensitivity of the security i's return to fluctuations in the common factors $\tilde{\delta}_{1t}$, $\tilde{\delta}_{2t}, \ldots, \tilde{\delta}_{Kt}$, respectively. Under the assumptions of the A.P.M. mathematical formulae which represents such coefficients cannot be derived.

However, if in addition one assumes that :

(a) The factors $\tilde{\delta}_{1t}, \tilde{\delta}_{2t}, \ldots, \tilde{\delta}_{Kt}$ are independent with each other .

(b) The disturbance terms are independent with each of the common factors .

(c) The joint distribution of security returns is multivariate normal.

then mathematical formulae for the K-factor beta coefficients can be derived.

Indeed if $\tilde{\delta}_{\xi t}$ is a common factor affecting the security returns then equation (4.2) implies :

$$\mathrm{Cov}(\tilde{R}_{it}, \tilde{\delta}_{\xi t}) = \mathrm{Cov}(r_i + b_{i1}\tilde{\delta}_{1t} + \ldots + b_{i\xi}\tilde{\delta}_{\xi} + \ldots + b_{iK}\tilde{\delta}_{Kt} + \tilde{e}_{it}, \tilde{\delta}_{\xi t})$$

Making use of the assumptions (a) and (b) the last equation becomes

$$\mathrm{Cov}(\tilde{R}_{it}, \tilde{\delta}_{\xi t}) = b_{i\xi}\sigma_{\delta_\xi}^2$$

Therefore

$$b_i = \frac{\mathrm{Cov}(\tilde{R}_{it}, \tilde{\delta}_{\xi t})}{\sigma_{\delta_\xi}^2} \qquad (5.1)$$

It is evident that equation (5.1) has the same form as beta in the C.A.P.M. However, equation (5.1) is derived with the help of three additional assumptions which are not required by the A.P.T. These assumptions are only necessary to give to the A.P.M. an empirical content (see also section 6.1.2).

(14) The market portfolio cannot be identified . Hence the C.A.P.M. cannot be tested empirically.
The factors that generate security returns cannot be identified. However, there are multivariate analysis techniques which help us to find the number of factors that influence security returns. Hence the A.P.M. may be tested. There are studies which claim that provided tests of the A.P.M. : Gehr, (1975), Roll and Ross (1980), Chen (1982), Reinganum, Hughes (1982) and Johnson (1981). But these tests are incomplete because they left untested the assumptions required to ensure that the A.P.M. can be tested unambiguously.

The previously stated comparison between the C.A.P.M. and the A.P.M. indicates that the models are not only quite distinct, but the latter cannot be regarded as an extension of the former.

Ross (1977) argued that if the 1-factor generating model of security returns is used, where the factor is the market portfolio, the C.A.P.M. can be proved. This proof is based on the extra assumption of the existence of "no arbitrage" condition in the market as defined by Ross. As noted earlier the C.A.P.M. can be proved even if this assumption is not fulfiled. Without this assumption the argument of Ross does not produce the C.A.P.M. However, Ross (1977) stated :

> "... but it should be emphasized that in a strict sense the underlying assumptions of the arbitrage theory and mean-variance theory are distinct. On purely theoretical grounds, then, it cannot be asserted that mean-variance theory is a special case of arbitrage theory..." (p.206)

The differences (2), (4), (6), (12) and (13) between the C.A.P.M. and the A.P.M. can also be recognised when the S.R.R.E.L.R. and the A.P.M. are compared. In this case it can be further observed the following :

(1) Since generally expectations are unobservable the S.R.R.E.L.R. alone provides no empirical hypothesis. To test it one may require an 1-factor return generating model, whose factor is specified.
The A.P.M. embodies a return generating model.

(2) To test the S.R.R.E.L.R. it is firstly estimated for each security in the sample the beta coefficient by using both the returns on the security and the returns on the market portfolio. To test the A.P.M. it is firstly estimated the factor beta coefficients for the securities in the sample by using the covariance (correlation) matrix of returns.

The conclusions about the relationship between the S.R.R.E.L.R. and the A.P.M. are the same as those regarding the relationship between the C.A.P.M. and the A.P.M.

5.3 Market Model versus K-Factor Security Returns Generating Model

Comparing the market model and the K-factor security returns generating model it is evident that ;

(1) The market model is heavily dependent on the assumption of the bivariate normality between each security's return

and the market portfolio's (market proxy's) return. Under this assumption the disturbance terms cannot be assumed to be independent. In addition this assumption implies that the market portfolio's (market proxy's) return is independent of the disturbance term for each security.
The K-factor security returns generating model places no restriction on the joint probability distribution between each security's return and the factors. This model assumes, however, that there are not dependencies among the disturbance terms. The K-factor security returns generating model does not need, in principle, the assumption of independence between factors and disturbance terms.

(2) Theoretically both are assumed to be static (one-period) models.

(3) The market model assumes that there exists a specific factor that generates security returns, namely the market portfolio (market proxy).
The K-factor security returns generating model assumes that there are K underlying factors that generate security returns, but it does not specify these factors.

5.4 Capital Asset Pricing Model versus Arbitrage Pricing Model : Some Common Misunderstandings

In spite of the clear differences between the C.A.P.M. and the A.P.M. it still appears that there is a certain degree of confusion in the academic literature about the two models.

Schalheim and DeMagistris (1980) for example, attempt to test the C.A.P.M. in a fashion similar to Fama and MacBeth (1973). The difference between their procedure and that of Fama and MacBeth is in the method that they employed to estimate the cross-sectional's regression coefficients. However, some of their conclusions are misleading. Indeed, they stated :

> "In light of the criticisms of the C.A.P.M. by Roll,the empirical tests in this paper may really be a test of the Arbitrage Pricing Model as derived by Ross. The form of the Arbitrage Pricing Model that coincides with the C.A.P.M. is the single-factor model". (p.65)[1]

Schalheim and DeMagistris used an observable market proxy to test a risk-return exact linear relationship and they asserted that they tested the 1-factor A.P.M. But the A.P.M. does not specify the underlying factor (s) behind the premium(s), while the C.A.P.M. (S.R.R.E.L.R.) specifies the market portfolio's (market proxy's) return. For this reason, when the A.P.M. is tested techniques of factor analysis are used, which do not depend implicity on the underlying factor(s).

Furthermore, they noted that the 1-factor A.P.M., where the single factor is the market portfolio, is equivalent to the C.A.P.M. However, it was explained previously that behind these models there are different assumptions and hence the two models are distinct.

---

1. Although they took into consideration Roll's criticisms they concluded that "Thus, our results substantiate the robustness of the F.M. conclusions in support of the C.A.P.M." (p.64).

Weston (1981) agreed with Schalheim's and DeMagistris' conclusions by saying :

> "From equation (5), Schalheim and DeMagistris also summarize from the Ross "Return, Risk and Arbitrage" paper the two-parameter or Black model..." (p.8).

Since Roll (1978) casts doubts on Jensen's portfolio performance measure within a C.A.P.M. framework, Pea snell, Skerratt and Taylor (1979) tried to interpret Jensen's procedure within an arbitrage pricing theory framework. Unfortunately, their argument may be criticized since they misunderstood the relationship between the C.A.P.M. and the A.P.M. In fact they stated :

> "The role of the C.A.P.M. in the arbitrage approach to evaluating fund performance is clearly coincidental - more accurately a C.A.P.M. based approach gives the same results in the single factor case. Where there is more than one systematic element (factor) in security returns the C.A.P.M. is misleading... And to the extent that the C.A.P.M. is an imperfect arbitrage model for adjusting for risk". (p.386).

> "...Furthermore, C.A.P.M. - based tests can be viewed as special cases of the arbitrage approach..." (p.387).

Their argument that the C.A.P.M. is inconsistent with a multi-factor model cannot be justified. The C.A.P.M. may be consistent with a multi-factor model. This is also supported by the fact that the market model does not assume interdependencies among the disturbance terms for different securities.

Also their conclusion concerning the tests of the C.A.P.M. as a special case of the arbitrage approach receives the same criticisms as Schalheim and DeMagistris' conclusion. The same criticisms regarding the work of Peasnell, Skerratt and Taylor were also made by Appleyard, Strong and Walker (1982). In addition they correctly pointed out that Peasnell, Skerratt and Taylor's arbitrage procedure for testing the mutual fund performance, using the Jensen measure, was based upon the presence of arbitrage opportunities. This is firstly inconsistent with equilibrium situations in the capital market and secondly it violates one of the major assumptions on which the A.P.M. is heavily based.

The relationship between the C.A.P.M. and the A.P.M. was also misunderstood by Langetieg (1978). According to Langetieg :

"In the special case where a security's return generating process depends on only a market factor and a second uncorrelated factor, Ross' no arbitrage model reduces to Black's zero-beta model. If a riskless asset also exists, Ross' no arbitrage model reduces to the Sharpe - Lintner model (p.368).

Next, Chen (1981) noted that :

"It is immediately apparent from (1) and (5) that the market model (see Fama (10) p.37) is a special case of the A.P.T."(p.6). [1]

and

"An alternative proof of the C.A.P.M. is to note that the

1 His equation (1) is the K-factor security returns generating model,while his equation(5) is the A.P.M.

multivariate normal distribution assumption implies via the above analysis, the market model (i.e. equation (1) with K=1, $\tilde{\delta}_1=\tilde{r}_m-E_m$) hence equation (6) " (footnote 7 p.31)[1].

Moreover he asserted :

"Another model that is consistent with (5) is the Kraus and Litznberger (23) Skewness preference model. By taking $\tilde{\delta}_1=\tilde{r}_m-E_m$ and $\tilde{\delta}_2=(\tilde{r}_m - E_m)^2$, we obtain their equation (6) ((23),p.1090) " (p.8)[2].

The security return generating model is a linear model, so it cannot be taken $\tilde{\delta}_2=(\tilde{r}_m-E_m)^2$. Even if one could take $\tilde{\delta}_2=(\tilde{r}_m-E_m)^2$ the expected return of $\tilde{\delta}_2$ is not equal to zero, as the assumptions of the security return generating model require.[3]

Finally, Copeland and Weston (1983) stated :

"If Eq.(7.45)[4] is interpreted as a linear regression equation (assuming that the vectors of returns have a joint normal distribution and that the factors have been linearly transformed so that their transformed vectors are orthogonal) then the coefficients, b_{ik}, are defined in exactly the same way as beta in the capital asset pricing

---

1. His equation (6) is :

$$E_p = \lambda_o + \frac{E_m - \lambda_o}{\sigma_m^2} \operatorname{Cov}(\tilde{r}_p,\tilde{r}_m)$$

2. (a) His equation (5) is $E_p=\lambda_o + (E-\lambda_o)V^{-1}\operatorname{Cov}(\tilde{r}_p,\tilde{r})$

(b) The equation (6) of Kraus and Litznberger is the ex-post form of the three moments C.A.P.M. and it is a quadratic return-risk relationship.

3. If $E(\tilde{\delta}_2) = E(\tilde{r}_m-E_m)^2=\sigma_m^2=0$ then the market portfolio (market proxy) should be riskless, which is totally impossible.

4. Their equation (7.45) is the A.P.M.

model...Hence, the C.A.P.M. is seen to be a special case of the A.P.T...." (p.214).

Here also there exists a misunderstanding concerning the relationship between the C.A.P.M. and the 1-factor A.P.M. As it was explained previously (see section 5.2) the factor beta coefficients of the security returns generating model can be described by mathematical formulae only if some additional conditions are assumed. But the theoretical derivation of Ross' A.P.M. does not require these additional assumptions. Thus, the conclusion of Copeland and Weston about the relationship between the C.A.P.M. and the 1-factor A.P.M. is incorrect.

5.5 Using the Arbitrage Pricing Model for Empirical Implementations

The A.P.M. was introduced in the literature as an alternative model to the expected return - standard deviation C.A.P.M. Since Roll exposed the fundamental problems regarding the tests of the C.A.P.M. substantial interest has focused on Ross' A.P.M. The A.P.M. rests upon a much simpler set of assumptions than the C.A.P.M. and it has the merit of avoiding the criticisms of the C.A.P.M.

Despite this fact the A.P.M. has a major disadvantage, namely it does not specify the identity of the factors which influence security returns. Without identifying the factors on which the A.P.M. is based, it is difficult to see how the A.P.M. can be used for empirical implementations .

Previous empirical work on the A.P.M. has relied on techniques of factor analysis, where a set of security returns is analyzed in order to generate a number of unobservable factors that affect security returns. The results showed that there does exist a small number of underlying factors determining the returns on securities.

What then do these findings tell us about the practical implications of the A.P.M. ?

(1) Can the A.P.M. be used as a benchmark to distinguish profitable from unprofitable investments ?

(2) Can the A.P.M. be utilized to incorporate risk into capital budgeting decisions ?

(3) Can the A.P.M. be employed to measure portfolio performance ?

(4) Can the A.P.M. be used for predictive purposes ?

Unfortunately these findings tell us nothing, because the empirical assumptions required to ensure an unambiguous test of the A.P.M. have been left untested.
If for example the factors which affect the security returns are not the same through time then the A.P.M. cannot be used for making predictions.
This is likely to happen since a factor can be found which has an important influence on security returns in one period but an unimportant influence on returns in the next period (a factor associated with a political crisis, oil crisis, etc.). Even if the same factors are relevant during various time periods there still exist some doubts about the

predictive usefulness of the A.P.M.

Friend (1981) expressed his opinion about this point as follows :

"Moreover, in the absence of an explicit description of the factors show that they can be used for attempted predicted purposes, there seems to me to be danger that statistically estimated factors may represent statistical artifact, resulting perhaps from our inability to specify rigorous significance tests." (p.352)

Chen (1981) listed a number of applications of the C.A.P.M. which can follow under the A.P.M. Theoretically, it is possible to prove these results. However, before any definite answer can be given concerning these applications, the assumptions which ensure an unambiguous test of the A.P.M. must be empirically verified (this point will be examined in greater details in Chapters 10 and 11).

There is a considerable body of study concerning either the theoretical and the empirical validity of the A.P.M. or some extensions of the A.P.M., but to my knowledge, as yet no study has been undertaken to examine empirically all the assumptions which ensure that the A.P.M. can be tested unambiguously using time series data. If the A.P.M. cannot be used for empirical implementations then its introduction to the literature as an attractive alternative to C.A.P.M. is rather questionable.

5.6 Conclusions

The expected return-standard deviation C.A.P.M. (S.R.R.E.L.R.) and the A.P.M. are two distinct models and neither is a special case of the other. Also, the market model is distinct from the K-factor security returns generating model.
The importance of verifying the assumptions which ensure an unambiguous test of the A.P.M. becomes apparent when topics involving its applications are approached.

CHAPTER 6

ARBITRAGE PRICING MODEL : THE EMPIRICAL EVIDENCE

The A.P.M. has been proposed as a testable alternative to the C.A.P.M. Therefore it has received recently widespread attention . Several empirical investigations concerning the A.P.M. have been presented in the literature. This chapter reviews the empirical studies of the A.P.M. performed by Gehr (1978), Roll and Ross (1980), Chen (1982), Reinganum (1981), Hughes (1982), Johnson (1981), Gibbons, (1981) , Kryzanowski and Chau (1982) and Dhrymes,Friend and Gultekin(1982).

6.1 The Empirical Evidence of the Arbitrage Pricing Model

There are several empirical studies of the A.P.M. performed in the time domain mainly for the U.S. market. These studies can be classified into two categories :

(1) Those which tested the empirical validity of the A.P.M. by examining the consistency between the observed data and the A.P.M. ; studies of this type were conducted by Gehr, Roll and Ross, Chen, Reinganum, Hughes and Johnson.

(2) Those which tested some of the assumptions of the A.P.M. in order to ensure that the A.P.M. can be unambiguously tested using time series data. Studies in this area were offered by Johnson, Gibbons , Kryzanowski and Chau and Dhrymes,Friend and Gultekin .

6.1.1 Procedure for Testing the Arbitrage Pricing Model

To test the A.P.M., all the previously mentioned studies employed a basic methodology that consisted of two steps.

STEP 1 Factor analysis techniques were employed to estimate the sensitivities of security or portfolio returns to the movements in the common factors (factor loadings).

STEP 2 Tests of the validity of the cross-sectional relationship between security or portfolio average returns and factor loadings, estimated from step 1, were constructed.

6.1.2 Transformation of the Arbitrage Pricing Model into a Testable Relationship

Besides the assumptions which embody the theoretical form of the A.P.M., there are some additional assumptions which give to it an empirical content.

The theoretical form of the A.P.M. can be set up as a relationship which applies to an individual investor. Hence, in order to transform it into a testable relationship one has to assume that all investors in the market have homogeneous beliefs on the security expected returns and factor beta coefficients.

There are also other assumptions necessary for the multivariate statistical procedures employed to test the A.P.M. Since factor analysis methods are primarily used

to estimate the factor loadings from time-series data on security or portfolio returns, the assumption of the particular factor analytic method are required.
For example, if Rao's factor analysis or maximum likelihood factor analysis is employed then the following has to be assumed:

(1) The joint distribution of security returns is multivariate normal and intertemporally stationary.

(2) The covariance or correlation matrix of security returns is non-singular.

(3) The factors $\tilde{\delta}_{1t}, \tilde{\delta}_{2t}, \ldots, \tilde{\delta}_{Kt}$ are independent with each other and each factor is independent with the disturbance terms of the security returns generating model.

These assumptions are also needed in tests for statistical inference. For example, the test of the goodness of fit of the factor model.
The above mentioned assumptions are necessary for the examination of the empirical validity of the A.P.M., but they are not required by the underlying theory of the A.P.M.

6.2 Tests of Gehr

The purpose of Gehr (1978) study was to establish whether there exists a multi-factor security returns generating model and, if so, what the number of pricing factors is.

The data he utilized is shown in Table 6.1. His test procedure was divided into two steps :

STEP 1 He grouped the entire sample period into three subperiods of equal length and he employed factor analytic techniques for each sample using the entire period and each subperiod.

STEP 2 He tested whether the mean returns on the indices were related to the beta coefficients produced by regressing the realized returns on the indices against the factors emerged using the 41 industry stocks.

For the first step factors corresponding to eigenvalues greater than one or less but very close to one were selected. His findings indicated that in the case of the indices there are two, or at most three, common factors that explain a large but no predominant portion, of the variance on indices. He also deduced for the case of the stocks that there are at least two factors having influence on stock returns.

For the second step of his study, he initially regressed the realized returns on the indices, against the factors created using the excess returns on the 41 industry stocks. The slope coefficients obtained from his first regression were then used as the independent variables in a regression equation which had the mean return on each industry index as a dependent variable. His empirical evidence indicated only one significant factor in the pricing relationship.

Table 6.1 Gehr's data

Source:	Center of Research in Securiry Prices Graduate School of Business University of Chicago Monthly Returns File
Sample Period Length:	30 years
Sample Size per Index:	360 monthly returns
Sample Size per Stock:	360 monthly returns
Constructive Criteria of the Industry Indices:	(i) Each index must contain at least five companies in a two digit industry group. (ii) A stock is included in the index if it has remained in the same industry for the entire sample period.
Selectoin Criteria of the Stocks:	(i) Each stock must remain in the same industry for the entire sample period. (ii) Each stock must be different from the stocks contained in the indices.
Number of Constructed Industry Indices:	24
Number of Selected Stocks:	41

Summing up, the general conclusion of Gehr's study are stated as follows :

(1) There exist two or three factors explained a large portion in variation in returns.

(2) There exists only one significant factor in the pricing relationship.

6.3 Tests of Roll and Ross

The objective of Roll and Ross' (1980) study was to investigate the existence of a multi-factor security returns generating model and the significance on the factors in the pricing relationship.

Their data are presented in Table 6.2. In order to test the A.P.M. they initially followed two steps :

STEP 1 They listed the 1260 securities into alphabetical order and they generated 42 groups of 30 individual securities. Then, for each group, they employed factor analysis techniques to estimate the factor loadings and then they performed a testing hypothesis about the number of factors.

STEP 2 They tested the following null hypothesis :

H_o: there exist non-zero constants $E_o, \lambda_1, \lambda_2, \ldots, \lambda_K$ such that

$$E_i - E_o = \sum_j \lambda_i b_{ij} \qquad \text{for each } i$$

Table 6.2 Roll and Ross' data

Source:	Center of Research in Security Prices Graduate School of Business University of Chicago Daily Returns File
Sample Period:	July 1962 - December 1972
Maximum Sample Size per Security:	2619 daily returns
Selection Criterion:	Each security must be listed on the New York or American Exchange on both 3 july 1962 and 31 December 1972.
Number of Selected Securities:	1260 (42 groups of 30 each)
Basic Data Unit:	Returns adjusted for all changes and including dividents, if any, between trading days.

Source : Roll and Ross(1980) , Table I , p.1086 against the alternative hypothesis :

H_1: there exist no non-zero constants $E_o, \lambda_1, \lambda_2, \ldots, \lambda_K$ such that

$$E_i - E_o = \sum_j \lambda_i b_{ij} \qquad \text{for each } i$$

The number of groups was chosen to be large in order to improve the statistical power of the tests. For the first step for each group the covariance matrix was computed and a maximum likelihood factor analysis was performed on this matrix. In order to test the hypothesis about the number of factors, they used the likelihood ratio principle. Their results showed that 32 groups out of 42 had at least an even chance that five factors were enough (see Roll and Ross, table II, p.1088).

For the second step the significance of the constants $E_o, \lambda_1, \ldots, \lambda_K$ was tested by employing the following multivariate cross-sectional regression :

$$\bar{r}_t = \hat{B}\lambda_t + \xi_t \qquad (6.1)$$

where

$\bar{r}_t$ = a (30 x 1) column vector of security excess mean returns.

$\hat{B}$ = a (30 x 5) matrix of factor loadings estimated in step 1.

λ_t = a (5 x 1) column vector with entries the risk premiums on factors.

ξ_t = a (30 x 1) column vector of security disturbances, having zero means and being dependent of one another.

Initially it was assumed λ_o= 6 percent. Their findings made them conclude that three or four factors were "priced". Next the constant λ_o was estimated by using the regression equation (6.1). In this case they found that there exist a small number of groups which produce at least three or four factors. Roll and Ross deduced that these findings were probably due to the incorrect choice of the zero beta return λ_o.

In addition they examined empirically :

(a) The A.P.M. against a specific alternative.

(b) The existence of a constant intercept across groups.

For the first test the following cross-sectional regression was run :

$$R_j = \lambda_{og} + \lambda_{1g} b_{1j} + \ldots + \lambda_{5g} b_{5j} + \lambda_{6g} S_j + \xi_j \qquad (6.2)$$

where

$j = 1,2,\ldots,30.$

$g = 1,2,\ldots,42.$

R_j= the arithmetic mean return on security j over the sample period.

b_{Kj}= the security j's loading on factor K.

$\lambda_{og},\ldots,\lambda_{6g}$ = the regression coefficient.

S_j = the individual security j's total standard deviation of daily returns during the sample period.

ξ_j = the security j's specific disturbance with zero mean and a non-zero variance.

In this case they found that in 19 groups out of 42 the estimated coefficient λ_{6g} was statistically significant at the 95% level of confidence. An F-test was also used to test the significance of the overall linear relation given by equation (6.2). Their findings indicated that the overall linear regression represented by equation (6.2) was significant in only 12 groups out of 42. These results made them doubtful about the validity of the A.P.M. However, before starting to conclude anything about the validity of the A.P.M., they examined the distribution of security returns. This examination was made because skewness can lead to spurious correlations between security mean return and security standard deviation.[1]
Indeed it was found that 1213 securities out of 1260 were positively skewed. In order to resolve many of the statistical problems produced by skewness they adopted the following procedure :

(I) Daily observations 3, 9, 15, 21,... were used and five factor loadings $b_{1j}, b_{2j}, \ldots, b_{5j}$, were estimated for each security in each of the 42 groups.

---

1. Miller and Scholes (1972, p.72) concluded that if there is skewness the cross-sectional regression will show a correlation between the mean return and the "own" standard deviation of return even though there is no such a correlation.

(II) Daily observations 5, 11, 17, 23,...were taken into consideration and the standard deviation of return was computed for each security.

(III) Daily observations 1, 7, 13, 19, ...were utilized and the following cross-sectional regression was run for each group g :

$$R_{jt} = \lambda_{ogt} + \lambda_{1gt} b_{1j} + \ldots + \lambda_{5gt} b_{5j} + \lambda_{6gt} S_j + \xi_{jt}$$

where j = 1,2,...,10.

On the basis of the results Roll and Ross deduced that in only 3 out of 42 groups the estimated coefficient λ_{6g} was statistically significant at the 95% level of confidence. Consequently they supported the A.P.M.

To examine the existence of a constant intercept, they tested, by using Hotelling T^2- statistic, the following hypothesis :

$$H_o: E(\lambda_{o,g-1,t} - \lambda_{o,g,t}) = 0 \quad ,g = 2,4,\ldots,38$$

Their results constituted evidence to conclude that the intercept terms were not different across groups.

Roll and Ross' general conclusions emerged from their tests can be summarized as follows :

(1) There are five statistically significant factors having influence in daily security returns.

(2) Three or four factors are "priced" in the market.

(3) The constant intercept of the A.P.M. is the same across different groups.

6.4 Tests of Chen

The purpose of Chen's (1982) study was to validate empirically the A.P.M.

The data used is presented in Table 7.3. His initial test of the A.P.M. consisted of two steps :

STEP 1 He used his own techneque and he estimated the factor loadings for all securities in each sample.

STEP 2 He tested the following null hypothesis :

H_o : there exist non-zero constants $\lambda_o, \lambda_1, \dots, \lambda_5$ such that

$$r_i = \sum_j \lambda_j b_{ij} \quad \text{for each } i$$

against the alternative hypothesis:

H_1 : there exist no non-zero constants $\lambda_o, \lambda_1, \dots, \lambda_5$ such that

$$r_i = \sum_j \lambda_j b_{ij} \quad \text{for each } i$$

If one divides the number of securities into groups, there is no guarantee that the factors of one group are the same as the factors of another group. Therefore he argued

Table 6.3 Chen's data

Source:	Center of Research in Security Prices Graduate School of Business University of Chicago Daily Returns File
Sample Period:	1963-1978 inclusive. The entire period is divided into four-subperiods: I. 1963-66,II.1967-70,III.1971-74, IV.1975-78.
Selection Criterion:	All the securities that do not have missing data during each subperiod and whose average daily return in absolute value is less than 0.01 to eliminate outliers.
Number of Selected Securities:	Subperiod — Total Sample I — 1064 II — 1522 III — 1580 IV — 1378
Basic Data Unit:	Return adjusted for all capital changes and including dividends.

Source : Chen(1982), Table 1 .

that it would be preferable if all the factor loadings corresponded to the same set of common factors. For this purpose he stated and proved the following :

Suppose N assets are given, having returns $\tilde{R}_i$, where $i = 1,2,\ldots,h+1,h+2,\ldots,N$. Assume that the returns on the first h + 1 assets are linearly independent. If one knows :

(i) The factor loadings of the h + 1 linearly independent assets, and

(ii) The Cov $(\tilde{R}_p, \tilde{R}_j)$, where $p = h+2,\ldots,N$, $j = 1,2,\ldots,h+1$, then he can determine uniquely the factor loadings of the assets $h+2,\ldots,N$.

Before attempting to apply this result, he selected for each subperiod the first 180 stocks in the sample and he computed their sample covariance matrix. Next the first 10 factor loadings for each of these stocks were estimated via maximum likelihood factor analysis. As a next step a linear programming procedure was utilized and 5 portfolios of equally weighted securities were created. Lastly, making use of the previously mentioned result, he estimated the first 5 factor loadings for every stock in each subperiod. He selected 5 common factors since other previous empirical studies confirmed that the number of factors is probably not greater than 5 (e.g. Roll and Ross (1980)).

In the second step the significance of the constants $\lambda_0, \lambda_1, \ldots, \lambda_5$ was tested by running for each subperiod the following multivariate cross-sectional regression:

$$r_i = \lambda_o + \lambda_1 b_{i1} + \ldots + \lambda_5 b_{i5} + \varepsilon_i$$

where

r_i = the mean return on security i computed on the even days of each subperiod.

b_{iK} = the factor loading of security i on factor K, K = 1,2,...,5.

λ_j = the excess return of the factor K over the mean return on a portfolio that is orthogonal with each factor.

ε_i = a disturbance term with zero mean and a non-zero variance.

His results showed a significant F - statistic at least at the 1-level of significance for every period (Chen(1982), Table 3A).

He also tested the A.P.M. against the "own variance" effect. His test method was different from the method used by Roll and Ross for the same purpose. His method can be briefly explained as follows :
For each subperiod the variance of each security was computed by using even days divisible by six. Then for each subperiod, all the securities were divided into two groups. One group was comprised of securities whose "own variance" was above the medium variance and the other was comprised of securities below. The next stage was to form two portfolios from the

groups such that :

(i) Each security, say i, in the portfolio has a weight x_i, where $x_i \geqslant 0, x_i \rightarrow 1/N$, N is the number of the securities in the portfolio.

(ii) Both portfolios have the same factor loadings.

Since one of the portfolios contains securities with high variance and the other those with low variance, it is clear that the A.P.M. should be valid if both portfolios had the same expected return.
The results of this test led Chen to argue that the "own variance" is not "priced". This implies that the expected security returns can only be affected by that risk which cannot be eliminated by portfolio diversification.

Therefore Chen concluded that the A.P.M. is a reasonable model for explaining cross-sectional variations in security returns.

6.5 Tests of Reinganum

Reinganum (1981) tested whether a parsimonious A.P.M. could explain the differences in average returns between small firms and large firms.

Table 6.4 describes his data. His procedure for testing the A.P.M. was divided into two steps :

Table 6.4 Reinganum's data

Source:	Center for research in Security Prices Graduate School of Business University of Chicago Daily Returns File
Sample Period:	1963 - 1978
Selection Criteria:	(i) The initial trading date for the security must be before the beginning of the year. (ii) The last trading date must be after that year (except for 1978). (iii) During the calendar year the security needs at least one hundred one-day returns. (iv) Year-end common share and price data has to be available in order to compute the market value of the common stock .

STEP 1 He estimated in year t - 1, t = 2,3,...,16, the factor loadings for all the securities in the sample by using the technique of Chen [1] and he formed control portfolios - that is portfolios containing securities with similar factor loadings. Then in year t, t = 2,3,...,16, he computed excess security returns by subtracting the daily control portfolio returns from the daily security returns.

STEP 2 He ranked securities with regard to the market value of their common stock at the end of year t - 1, t = 2,3,...,16 and he formed according to this ranking 10 market value portfolios. He then tested the following implication of the A.P.M. If the A.P.M. holds, then the average excess return on the 10 portfolios should be equal.

In step 1 for each year the number of securities was divided into thirty portfolios and the (30 x 30) covariance matrix was estimated. The factor analysis was then performed on this matrix and the technique of Chen to estimate the factor loadings was ued. Next control portfolios were constructed by taking into consideration three, four and five factors. Lastly for each t, t = 2,3,...,16, the excess security returns were calculated.

For step 2 he ranked securities on the basis of the market value of the common stock at the end of the year t - 1, t = 2,3,...,16 and he divided them into 10 portfolios.

---

1. See Section 6.4.

For each portfolio the t^{th} year excess return was computed by applying equal weights to the t^{th} year excess security returns, $t = 2,3,\ldots,16$. Therefore if the A.P.M. was valid, then the excess returns on the market value portfolios should be equal to zero. His findings, however, indicated that the 10 average excess returns were not equal to zero.

Consequently he rejected the A.P.M. as a preferable alternative to the C.A.P.M.

6.6 Tests of Hughes

The objective of Hughes' (1982) study was to examine the validity of the A.P.M. and the existence of a constant intercept equal to the riskless rate of interest when a security's expected return is described in terms of the risk premiums on the factors.

Her data is summarized in Table 6.5. Her test concerning the A.P.M. comprised of two steps :

STEP 1 She divided the number of 220 securities into two groups and she factor analysed each group to estimate the factor loadings.

STEP 2 She tested the following null hypothesis :

H_o : there exist non-zero constants $\lambda_o, \lambda_1, \ldots, \lambda_K$ such that :

$$r_i = \lambda_o + \sum_j b_{ij} \lambda_i \qquad \text{for each } i$$

against the alternative hypothesis:

Table 6.5 Hughes' data

Source:	Wood Gundy Ltd.(Canada)
Sample Period:	January 1971 - December 1978
Sample Size per Security:	120 monthly returns
Selection Criterion:	Each security must be listed on the Toronto Stock Exchange for the entire sample period.
Number of Selected Securities:	220
Basic Data Unit:	Returns were adjusted for stock splits and dividends.

H_1 : there exist no non-zero constants $\lambda_o, \lambda_1, \ldots, \lambda_K$ such that :

$$r_i = \lambda_o + \sum_j b_{ij} \lambda_i \quad \text{for each } i$$

For step 1 two groups of 120 securities were formed and the "minimum residuals" (Minres) method was used to estimate the factor loadings and the factor scores for each group. The following two justifications were behind the size of her groups :

(i) Since in factor analysis the factors cannot be identified, there is no method to ascertain whether the factors of one group are identical to the factors of another group. Hence as the number of groups increases the problem of the intepretation of the results also increases. Thus the fewer groups used the fewer problems of interpretation occur.

(ii) The size of each group has to be smaller than the number of observations, since if the number of variable exceeded the number of observations the resulting covariance or correlation matrix should be singular.

Gibbons (1982) found that the correlation matrix of security returns was stationary through time and the covariance matrix was not intertemporally stationary. In view of Gibbons' results Hughes concluded that the correlation matrix should be factor analyzed. Therefore by using the Minres factor analytic method she extracted 12 factors from each correlation matrix. Next the significance of the constants $\lambda_o, \lambda_1, \ldots, \lambda_{12}$ was tested by adopting a

method introduced by Litzenberger and Ramaswany (1979). This method produces estimates of the constants $\lambda_o, \lambda_1, \ldots, \lambda_{12}$ equivalent to the generalized least squares estimators employed by Roll and Ross (1980). This method can be briefly described as follows :
For each constant $\lambda_o, \lambda_1, \ldots, \lambda_{12}$ can be estimated an unbiased and asympotically efficient estimator[1]. Such an estimator is the expected return of a minimum variance arbitrage portfolio whose investment proportions vector can be defined by the constrained minimization problem :

$$\text{Min} \quad X' U \, U' X$$

$$\text{Subset to} \quad b' X = C$$

where

U = a (110x110) summetric matrix with unknown diagonal elements $u_j, j = 1,2,\ldots,110$.

$b = (B, i_1)$ is a (110x13) matrix, where B is the (110x12) matrix of factor loadings and i_1 is the (12x1) unit vector.

C = a (13x1) column vector containing an entry equal to one and 12 entries equal to zero.

By solving for each group 12 constrained minimization problems, similar to that described above, she computed for each group 12 matrices of minimum variance arbitrage portfolio weights.

---

1. An estimator is called asymptotically efficient if it is an asympotically unbiased estimator having the smallest asympotic variance among a group of asymptotically unbiased estimators.

Then she estimated the constants $\lambda_o, \lambda_1, \ldots, \lambda_{12}$ by using the following equation :

$$\hat{\lambda}_{it} = \sum_j x_j^i \, r_{jt} - F_{it} \tag{6.3}$$

where

$i = 1,2,\ldots,12$.

F_{it} = the value of common factor i estimated via Minres factor analytic technique.

Finally making use of the student t-statistic she came to the conclusion that in each group three or four factors were "priced".

In addition to the A.P.M.'s test she empirically investigated the following hypotheses :

(a) There exists a coefficient λ_o which is constant for all securities and equal to the riskless rate of interest.

(b) The constants $\lambda_1, \ldots, \lambda_{12}$ estimated by utilizing one security group, can be used to explain the variation in security returns for the second group of securities.

For the first test she initially estimated for each group the intercept λ_o for each security in each month by using the following equation :

$$\lambda_{o,j,t} = r_{jt} - \sum_j b_{ji} \lambda_{it} - \sum_j b_{ji} F_{it}$$

where

$i = 1,2,\ldots,110$.

$t = 1,2,\ldots,120$.

λ_{it} is estimated with the aid of equation (6.3).

Then for each group the following null hypothesis was tested :

$$H_o : \bar{\lambda}_{o,1} = \bar{\lambda}_{o,2} = \cdots = \bar{\lambda}_{o,110} \quad [1]$$

by using Hotelling T^2 - statistic. Based upon her results she inferred that for each group there exists a constant intercept across securities. Furthermore she used Hotelling T^2 - statistic to test for each group the following null hypothesis :

$$H_o : \bar{\hat{\lambda}}_{o,1} = \bar{\hat{\lambda}}_{o,2} = \cdots = \bar{\hat{\lambda}}_{o,110}$$

where

$$\bar{\hat{\lambda}}_{o,j} = \bar{\lambda}_{o,j} - r_F \quad , j = 1,2,\ldots 110 .$$

r_F = the mean return on 90-day treasury bills or the mean return on 30-day banker's acceptances.

Her result revealed that the intercept term was equal to the risk free rate.

To test the second hypothesis she regressed monthly security returns for each firm in group A (B) on monthly estimates of $\lambda_{1t}, \ldots, \lambda_{12t}$ for group B (A). In the light of her results, she argued that the estimated $\lambda_{1t}, \ldots, \lambda_{12t}$ from one group of securities, explained a large proportion of variation in security returns for a second group of securities.

Her general conclusions may be summarized as follows :
(1) Three or four factors are "priced" in the market.

1 The bars indicate average values over her sample period.

(2) There exists a constant intercept for all securities and it is equal to the riskless rate of interest.

6.7 Tests of Johnson

The purpose of Johnson's (1981) study was to form orthogonal indices which can be utilized as inputs to a multi-factor model and to investigate the empirical validity of the A.P.M. using security indices. Johnson's study is the only study which verifies empirically the A.P.M. using U.K. data.

Johnson's data are briefly described in Table 6.6 below. Three of the most important steps in his work can be summarized as follows :

STEP 1 He examined different methods to form orthogonal indices (factors) which may be used as inputs to a multi-factor model.

STEP 2 He tested the A.P.M.

STEP 3 He tested the relationship between the number of factors which have a direct effect upon security returns and the size of groups being factored.

To form orthogonal indices he used two methods. Firstly,

Table 6.6 Johnson's data

Source:	London Share Price Database Graduate Business School University of London Monthly Returns File
Sample Period:	January/February 1971-June 1978
Number of Constructed Indices:	35
Number of Selected Securities:	1308
Sample Size per Index:	89 monthly returns
Sample Size per Security:	89 monthly returns

he analysed the data of 32 indices, via principal component analysis and he deduced that 6 homogeneous groups emerged from his analysis. Secondly, he regressed the returns of the 32 indices against the return of the F.T.A. index and he estimated the beta coefficient. Then he adopted the procedure of Black, Jensen and Scholes (1972) and he formed according to his beta values 7 homogeneous risk groups.

To examine empirically the validity of the A.P.M. he used as basic data the monthly returns on 33 indices. Then a factor analytic technique was adopted to estimate the factor loadings and statistical analysis techniques were employed to test the cross-sectional pricing conclusions of the A.P.M. By specifying the number of factors to be equal four he concluded that his results supported the A.P.M.

Johnson tested the relationship between the relevant number of factors and the group size by utilizing the sample of securities and employing two different procedures. Initially the whole number of securities was divided into 5 samples and for each security the beta coefficient was estimated by using the single-index model. Then each sample was divided into 9 subsamples according to the formed beta distributions. His reported evidence showed that the number of factors is not the same across various samples. Johnson also formed random portfolios comprised of twenty securities. Again his results indicated that different number of factors are assigned to each sample of twenty securities. Unfortunately Johnson did not state the method that he used to test the goodness of fit of the factor models.

The conclusions derived by Johnson can be summarized as follows :

(1) There are six or seven components that account for the total variance of a group of 32 indices.

(2) By using the data of the indices the A.P.M. cannot be rejected.

(3). The number of factors is not the same across different security groups.

6.8 Tests of Gibbons

Gibbons (1981) examined empirically the return generating process of the A.P.M. by utilizing stock and bond portfolios.

Table 6.7 below presents Gibbons' data. In his study Gibbons tested :

(1) The adequacy of a K-factor model for generating the portfolio returns and the relevance of industry factors having a direct influence upon portfolio returns.

(2) The intertemporal stationarity of the covariance and correlation matrices.

(3) Whether the number of factors is sensitive to the type of securities included in the sample.

Table 6.7 Gibbons' data

Source:	Center of Research in Security Prices Graduate School of Business University of Chicago Monthly Returns File
Sample Period:	January 1953 - July 1971
Sample Size per Security:	224 monthly returns
Selection Criteria:	(i) Each security must be listed continuously on the New York Stock Exchange for the entire sample period. (ii) Each security must maintain the same industry classification for the entire sample period.
Number of Constructed Industry Portfolios:	41
Number of Created Bond Portfolios:	9

To determine the relevant number of factors required to describe the covariance structure of the 41 stock portfolios he used the appropriate likelihood ratio technique.
His findings implied that more than 8 factors are needed to explain portfolio returns. His evidence contrasted with Roll and Ross' (1980) findings which showed that 4 to 5 factors may be adequate. Since each portfolio in his sample was constructed from securities belonging to the same industry, a particular industry's factor loading is non-zero for the security in that industry. Hence he tested for zero security factor loadings. His results made him deduce the inadequancy of the breaking down a security's or portfolio's return into a market and industry factor.

For his second test the entire sample period of the 224 months was divided into two equal subperiods. Then the appropriate likelihood ratio test was employed for the homogeneity of the covariance matrix across the two subperiods for the following cases :

(i) Given 41 stock portfolios.

(ii) Given 9 bond portfolios.

(iii) Given 41 stock and 9 bond portfolios.
His results rejected the intertemporal stationarity of the covariance matrix for each case. Gibbons also investigated empirically the homogeneity of the correlation matrix across the two subperiods for the above mentioned cases. This time he made use of a chi-square statistical test suggested

by Jennirch (1970). His evidence supported the intertemporal stationarity of the correlation matrix for each case. Therefore he concluded that for an empirical investigation of the A.P.M. the correlation matrix should be factor analyzed.

Finally, he used the same likelihood ratio technique, as he used in his first test, to determine the number of factors which affect :

(i) The return on the 41 stock portfolios.

(ii) The return on the 9 bond portfolios.

(iii) The return on the 41 stock and 9 bond portfolios.

In view of his evidence he concluded that analyzing together stock and bond portfolios additional factors, common to both groups, had influence on returns. These factors, however, were not found when he analyzed only one group of securities.

6.9 Tests of Kryzanowski and Chau

Kryzanowski and Chau (1982) empirically tested whether the number of factors is dependent upon the number of securities included in a group or upon the sample size in terms of time period.

The data they utilized are summarized in Table 6.8. Kryzanowski and Chau tested :

(1) The relationship between the number of factors that

Table 6.8 Kryzanowski and Chau's data

U.S. DATA	
Source:	Center of Reasearch in Security Prices Graduate School of Business University of Chicago Monthly Returns File
Sample Period:	January 1948 - December 1977 For each security six samples in terms of time periods were generated:I.1948-57, II.1958-67,III.1968-77,IV.1948-62 V.1963-77,VI.1948-77.
Selection Criterion:	Each security must be listed on the New York Stock Exchange continuously during the entire sample perion.
Number of Selected Securities:	550(11 groups of 50 each)
Basic Data Unit:	Returns adjusted for all capital changes and including dividend.
CANADIAN DATA	
Source:	Jones Heward and Cie (Canada)
Sample Period:	January 1962 - December 1971 For each security two samples in terms of time periods were generated: I.1962-66,II.1967-71
Selection Criterion:	Each security must be listed on the Torondo Stock Exchange continuously during the entire sample period.
Number of Selected Securities:	180 (3 groups of 60 each)
Basic Data Unit:	Returns adjusted for all capital changes and including dividend.

affect security returns and the sample size in terms of time periods.

(2) The relationship between the number of factors that affect security returns and the size of the group being factored.

For the first test both Rao and alpha factor analysis were used to determine in each of the six time intervals the relevant number of factors that related to security returns. Both Rao and alpha factor analytic techniques showed that the larger the sample size in terms of time periods, the simpler is the factor structure in terms of the number of factors associated with security returns. Therefore they concluded that, on average, the number of factors associated with security returns remained approximately the same across various samples of the same size and across various time intervals.

For the second task they randomly drew from the first group of securities four subgroups containing 10, 20, 30 and 40 securities respectively. To determine for each of the four subgroups the relevant number of factors that account for the security intercorrelations they also employed Rao and alpha factor analysis. Both factor analytic methods showed that the number of relevant factors increases with the group size.

Comparing the results with those of Roll and Ross (1980) they stated that their findings may be due to the use of a smaller sample size per security (360 observations against

1260 observations of Roll and Ross).

Finally, they applied the alpha and Rao's factor analytic methods to the three samples of Canadian securities. Their conclusions were similar to those derived by utilizing the eleven samples of U.S. securities.

6.10 Tests of Dhrymes, Friend and Gultekin

The objective of Dhrymes, Friend and Gultekin's (1982) study was to re-examine the evidence presented by Roll and Ross (1980) and point out some criticisms concerning their tests. Dhrymes, Friend and Gultekin utilized the data used Roll and Ross.
Three of the most important issues that they examined empirically are summarized as follows :

(1) The relationship between the number of factors determining the security returns and the group size being factored.

(2) The existence of multiple factors which generate security returns.

(3) The existence of a constant intercept across various groups which is equal to the riskless rate of interest.

For the first test they randomly drew from a group containing 90 securities 4(overlapping) subgroups containing 15, 30, 45 and 60 securities,respectively. Then by employing the maximum likelihood factor analytic technique they performed a testing hypothesis concerning the number of factors. Their findings indicated a positive relationship between the number of factors and the group size. In view of these findings they questioned Roll and Ross' result that security returns are determined by 5 factors. Dhrymes, Friend and Gultekin noted that a possible explanation of Roll and Ross' result may be due to missing daily return observations. According to Dhrymes, Friend and Gultekin :

> "In the RR sample, there are several securities with more than 800, and one with more than 1400, missing observations. Using these securities would have eliminated much more than half of the observations in joint tests." (Ft. 3, p.58)

The results of their second test showed that, in terms of explanatory power, a five-factor returns generating model is superior to a one-factor returns generating model.

Finally, they adopted Roll and Ross' methodology and they tested the existence of a constant intercept across various groups. According to their results they concluded that the intercept terms, on average, are equal across various groups. On the other hand, they tested the hypothesis that all intercept terms are equal to zero. Their empirical evidence revealed that, on average, the intercept terms are insignificantly different from zero for nearly all groups. This result made them to cast some doubts on the usefulness of the A.P.M.
Table 6.9 summarizes the conclusions of the empirical studies of the A.P.M.

Table 6.9 Summary results of the arbitrage pricing model's tests.

STUDY OF	COUNTRY	CONCLUSION(S) OF THE STUDY
Gehr	U.S.A.	Accept the A.P.M.
Roll and Ross	U.S.A.	(1) Accept the A.P.M. (2) There exists a constant intetcept across security groups.
Chen	U.S.A.	Accept the A.P.M.
Reinganum	U.S.A.	Reject the A.P.M.
Hughes	CANADA	(1) Accept the A.P.M. (2) There exists a constant intercept equal to the riskless rate of interest.
Johnson	U.K.	(1) It is possible to form orthogonal indices(factors) and use them as inputs in a multi-index model. (2) Accept the A.P.M. (3) The number of factors which affect security returns is not the same across different (nonovelapping) groups of securities.
Gibbons	U.S.A.	(1) The number of factors having influence in stock portfolio returns is larger comparing with this concluded by Roll and Ross. (2) In the A.P.M.'s tests the correlation matrix of returns should be factor analyzed. (3) The number of factors emerged by analyzing together stock and bond portfolios is larger than the number of factors emerged by analyzing only one group of portfolios.
Kryzanowski and Chau	1. U.S.A. and 2. CANADA	(1) In average, the relevant number of factors which require to describe the correlation stucture on securities returns remains approximately the same across variuos groups of the same size and across various time periods of equal or different sizes. (2) The relevant number of factors which affect security returns increases with the group size.
Dhrymes Friend and Gultekin	U.S.A.	(1) The relevant number of factors determining the returns on securities increases with the group size. (2) The intercept terms are equal across various groups,but they are, on average ,insignificantly different from zero.

6.11 Necessary Assumptions for an Unambiguous Test of the Arbitrage Pricing Model

It was pointed out earlier that the transformation of the theoretical A.P.M. into a testable relationship requires some additional assumptions. These asssumptions can be summarized as follows :

(1) The distributions of returns on securities are normal.

(2) The distributions of security returns are intertemporally stationary.

(3) The number of common factors which influence the security returns is the same across various security groups having different sizes and across different security groups having the same size.

(4) The number of common factors which affect the returns on securities is stable across various time periods for the same group of securities and across various time periods for different groups of securities.

(5) The same common factors generate the security returns in groups of different sizes and in groups of equal size.

(6) The same common factors influence the security returns across various time periods for the same group of securities and across different time periods for different security groups.

(7) The factor beta coefficients are intertemporally stationary.

If the previously mentioned assumptions were needed for the theoretical derivation of the A.P.M., the validity of the A.P.M. would have to be determined by testing its predictions and not its assumptions. However, these assumptions are necessary when the A.P.M. is empirically tested using time series data, but they are not required by the theory behind the model. Therefore the empirical verification of such assumptions is very important in order to determine whether or not time series security returns can be utilized to test unambiguously the A.P.M.

It can be also noted that the validation of these assumptions will not imply the validity of the A.P.M. Furthermore the violation of one or more of such assumptions will not constitute evidence against the A.P.M. It will simply show that the present statistical methodology cannot be used to provide an unambiguous test of the model.

Assumption 1 A large number of statistical tests are based upon the assumption of normality. Some of these tests are robust under normality (e.g. the maximum likelihood test for the goodness of fit of the factor model) and other are sensitive to violations of the normality assumption (e.g. a chi-square test for the homogeneity of two covariance matrices of security returns) . If the assumption of normality is violated the tests of the second group may produce biased results. Since an unambiguous verification of the A.P.M. requires the intertemporal stationarity of the covariance or correlation matrix and the statistical tests of such issues are very sensitive to departures from normality (see Mardla (1971)) it is necessary to perform a test for normality.

Gibbons performed two tests which are sensitive to violations of the normality (a chi-square test for the homogeneity of two covariance matrices of security returns and a chi-square test for the homogeneity of two correlation matrices of security returns). However, he did not examine empirically the normality assumption of the distributions of portfolio returns.

Assumption 2 The A.P.M. describes an approximate linear relationship between expected returns on securities and factor beta coefficients. Neither of these expected returns on securities nor the factor beta coefficients are directly observable. Consequently, the A.P.M. is tested with the aid of ex-post data. The use of ex-post data implies the substitution of ex-post distributions for ex-ante distributions. Thus it is necessary to assume that the security returns obey a stationary multivariate distribution during the sample period. Under this assumption realized distributions of security returns can treated as sample observations of the ex-ante joint distribution of security returns.

Furthermore, the K-factor security returns generating model and the A.P.M. are static (single period) models, although for testing purposes they are treated as if they hold intertemporally. This is a necessary condition because the validity of the A.P.M. is verified by utilizing time series data. There is no guarantee, however, from the arbitrage pricing theory that the joint distrubtion of security returns is stationary during the sample period.

Lastly the intertemporal stationarity of the security mean returns is compatible and gives strength to the assumption concerning the investors homogeneous beliefs on the security expected returns. As it was stated in Section 6.1.2 the latter assumption is necessary to make the A.P.M. suitable for empirical tests.

From the previous discussion it is clear that direct empirical tests of the intertemporal stationarity of the distributions of returns on securities are required before to empirically validate the A.P.M. in an unambiguous fashion.

The normality assumption is necessary for testing the second assumption because :

(1) It implies that testing the intertemporal stationarity distribution of security returns is equivalent with testing the intertemporal stationarity of the security mean returns and covariance matrix of security returns.

(2) Some of the tests require to verify the second assumption are sensitive to departures from normality (e.g.the chi-square test for the homogeneity of two covariance (correlation) matrices of security returns).

Gehr, Roll and Ross , Chen, Reinganum, Hughes and Johnson left unverified the assumption of the intertemporal stationarity distribution of security returns. Therefore there tests of the A.P.M. may be characterized as incomplete on this ground alone.

Only Gibbons tested the intertemporal stationarity of the covariance (correlation) matrix by utilizing portfolios. Although he adopted the appropriate multivariate statistical techniques his results reveal lack of statistical power, because he only used two groups of portfolios.

Lastly Hughes noted that she used the correlation matrix, because Gibbons concluded that the correlation matrix should be factor analyzed. However, the results obtained by utilizing data from different international stock markets do no necessarily coincide. Thus Hughes justification for using the correlation matrix is left behind the rigorous theory.

<u>Assumption 3</u> The theoretical derivation of the A.P.M. is based upon the existence of the security returns generating model. Such a model describes linearly the returns of a large group of securities in terms of K common factors and a specific factor. That is from the theoretical point of view one considers a large number of securities, say N, and he assumes a fixed number of K-factors having influence on security returns, where $K < N$.

The security returns generating model assumption is one of the most important assumption of the A.P.M. Hence it is necessary to verify empirically such an assumption and then proceed to examine the validity of the A.P.M.

The security returns generating model assumption has to be empirically examined for the following reasons :

(1) To estimate the number of factors explaining the variation in security returns. Such an estimation firstly shows whether there exist multiple factors which generate the returns on securities. Secondly it is very important when the empirical applications of the model are considered. The A.P.M. requires the set of securities to be large, i.e., large enough so as to assure the application of the law of large numbers. Hence the A.P.M. has to be tested by utilizing large samples of securities. But the joint analysis of a large number of securities becomes computationally impossible. Consequently, it is necessary to divide the entire sample of securities into different groups and factor analyze each group separately. However, if the number of factors determining the security returns is positively related to the group size, then by considering a large number of securities the number of factors could be so large that it severely inhibited the application of the model.

(2) Given the existing methodological procedure of testing the A.P.M., one also has to investigate the existence of a unique security returns generating model across the various groups. The importance of this point can be explained as follows :
Suppose the number of factors determining the security returns is positively related to the group size. Then an obvious question arises ; which is the appropriate group size that has to be used in order to test the A.P.M. unambiguously ? However, the A.P.T. does not provide any help in selecting the appropriate security returns generating model, because it assumes a unique generating model consisting of a fixed number of unobservable factors. Therefore by considering a given group size and producing a linear model that describes the security returns, there is no way to ascertain whether such a model is the unique model upon which relies the A.P.M. As a consequence the methodological procedure employed to test the A.P.M. does not necessarily result in tests of the model. In such a case it can be concluded that the A.P.M. cannot be tested unambiguously.

Although the assumption concerning the relationship between the number of factors and the group size is very important for an unambiguous test of the A.P.M. such an assumption left untested or partial tested.

Gehr, Roll and Ross, Chen, Reinganum and Hughes did not empirically verify such an assumption. However, each of these studies found the existence of multiple factors affecting the security returns.

Gehr used a rule of thumb and he selected only factors with eigenvalues greater than one or less but very close to one. But the eigenvalue-one rule of thumb is not a reliable criterion. According to Rummer (1970) :

> "Small data errors, applying one distributional transformation versus another, unequally skewed distributions, and various design decisions can shift the eigenvalues that are close to unity from above-one acceptability to below-one acceptability, and vice versa." (p.363).

Roll and Ross, Reinganum and Hughes tested the goodness of fit of the factor model.

Roll and Ross generated 42 groups of 30 individual securities and they concluded that five factors were sufficient in explaining the intercorrelations in daily returns. However, if the number of factors is positively related to the group size, then the methodological procedure used by Roll and Ross is inappropriate. This is true since the consideration of their whole sample of securities could produce more factors than those which they claimed to influence the security returns.

Reinganum generated 30 portfolios and he inferred that more than five factors were needed for adequate factoring. Roll and Ross and Reinganum utilized U.S. data, but they considered different sample periods. This implies that the number of factors is not the same across different time periods. However, Reinganum did not give any explanation concerning the comparison of his results with those of Roll and Ross.

Hughes formed groups of large size (110 securities) and she deduced that more than 12 common factors are needed to explain security returns for the Toronto Stock Exchange. She justified the large number of factors by notifying that Kryzanowski and Chau found for the Toronto Stock Exchange a positive relationship between the number of factors and the group size being factored. But Kryzanowski and Chau found that there were, on average, 18 relevant factors by factor analyzed groups of 60 securities. Therefore from her groups should be emerged a large number of factors and hence the empirical applications of the model for the Toronto Stock Exchange should be rather questionable. However, Hughes did not provide any discussion about this important point.

Chen did not test the goodness of fit of the factor model, but he decided to select five factors because Roll and Ross' results indicated that five factors are appropriate. However, if the number of factors is positively related to the group size, his choice to select the first five factors emerged from a group containing 180 securities is a grave error.

Lastly Johnson chose arbitrarily four factors determining the returns for the London Stock Exchange.

Gibbons only tested whether the number of factors is the same across different groups of portfolios, while Kryzamowski and Chau and Dhrymes, Friend and Gultekin empirically examined whether the number of factors is related to the group size. These studies reveal lack of statistical power because Gibbons used only three groups of portfolios, whereas Kryzamowski and Chau and Dhrymes, Friend and Gultekin utilized one group of securities and four (overlapping) subgroups of different sizes generated from this group.

Finally, Johnson used disjoint security groups (i.e. security groups which do not contain common securities) to test the relationship between the number of factors and the group size being factored.

However, it is preferable to utilize different groups of securities and generate from each group various security subgroups. In this case it is possible to compare the number of factors between :

(1) Groups of different sizes containing some securities in common.

(2) Groups of the same size which do not contain common securities.

(3) Groups of different sizes which do not contain common securities.

As it can be seen the testing design of Johnson is a special case of the previously mentioned testing design.

<u>Assumption 4 and Assumption 6</u> In principle the security returns generating model is a static model. A usual assumption made when a time series data is used to test the A.P.M. is that the security returns generated model holds in each required time interval (e.g. month, year) of the sample period. In this case except of the assumption regarding the intertemporal stationary distributions of returns is needed an additional assumption which is also necessary to make the A.P.M. suitable for empirical verifications. This assumption is that the common factors affecting the security returns remains unchanged across the various time intervals of the sample period.

Due to the identification problem of the factors,there is no good way to ascertain whether the factors having influence on security returns are replicable across various time periods. The only way to reject this assumption is to verify whether the number of factors affecting the security returns remains unchanged across variuos time periods for the same group of securities and across various time periods for different security groups. If the number of factors is not stable true time then maybe exist few common factors which are the same from period to period,while the entire set of the common factors is not replicable across various time periods.

There is also another major importance of examining the stability of the number of factors across various time periods. Indeed if the number of factors is not the same from period to period then there is no hope concerning the empirical implementations of the A.P.M. (see Section 5.5).

This important assumption has not been verified in the literature. Only Kryzamowski and Chau tested whether the number of factors remains unchanged across various time periods for the same group of securities (see Section 7.9) .

<u>Assumption 5</u> The discussion concerning this assumption is very similar to the discussion stated for the assumption 3. That is if there are not the same factors across different security groups then the A.P.T. does not provide the means in concluding which are the relevant factors affecting the security returns. Therefore the tests of the A.P.M. may be questioned because the relevant factor structure of security returns can not be identified (see also assumption 3).

The present assumption will not be valid if the number of factors which affect the security returns changes across various groups of different sizes and across various groups of the same size. Once more the present assumption was only partially tested.

Roll and Ross and Hughes tested only the existence of a constant intercept, equal to the risk free rate across different groups of the same size.

Chen and Reinganum used only one group of securities and they estimated the first five factor loadings. Then with the help of these estimated factor loadings they computed the first five factor loadings for every security in the sample. They concluded that this technique produces the same set of common factors across the securities in the sample. But the problem of the factors identification implies that empirically it is not known whether the five factors emerged from the first group are the most important factors having influence on the remaining security returns.

Assumption 7 One can proceed to test this assumption if the common factors affecting the security returns are the same across various time periods. Indeed if the previous assumption is valid then one has to test the present assumption before to apply the A.P.M. for predictive purposes. Furthermore, this assumption is a necessary condition for estimating the factor loadings and factor scores using a common factor analytic technique.

Lastly, the intertemporal stationarity of the factor beta coefficients is compatible and gives strength to the assumption regarding the investors homogeneous beliefs for the factor beta coefficients. As it was mentioned in Section 6.1.2 the latter assumption is required to transform the A.P.M. into a testable relationship.

The present assumption, however, has been left untested.

If the previously mentioned assumptions are not verified it seems dangerous to derive conclusions concerning the validity of the A.P.M., because some conclusions may be implied from the violation of one or more assumptions. It is evident that the results of Gehr, Roll and Ross, Chen, Reinganum, Hughes and Johnson would be more powerful and reliable if they first tested empirically such assumptions. Without verifying these assumptions their tests are not complete and they should be interpreted with caution.

Furthermore, Gibbons, Kryanowski and Chau and Dhrymes, Friend and Gultekin verified empirical some of these assumptions. However, their results should be statistically powerful if they utilized more groups of portfolios (securities).

Consequently, given the importance of these assumptions the the incompleteness of the previous tests the empirical part of this study is concerned with an investigation of such assumptions for the London Stock Exchange.

6.12 Conclusions

Since the theoretical formulation of the A.P.M., there have been a number of empirical tests concerning its validity and/or the verification of some basic assumptions required to ensure an unambiguous test of the model.

The first group of tests are incomplete, because they did not verify the basic assumptions of the A.P.M. The second group of tests reveal the lack of statistical power.

This study is concentrated upon an empirical investigation of the A.P.M.'s basic assumption for the London Stock Exchange.

CHAPTER 7

FACTOR ANALYTIC TECHNIQUES AND THEIR APPLICATION TO PORTFOLIO THEORY

Since the factor analytic techniques have been utilized by a number of studies in portfolio theory and since a large part of this empirical examination uses factor analysis, the present chapter briefly presents the theory of factor analysis. The chapter begins by introducing the factor analysis model. Next it provides a comparison between the principal component analysis and the factor analysis and it presents some criticisms concerning the factor analytic methods. Finally, it discusses the factor analytic techniques utilized in testing the A.P.T.

7.1 The Factor Analysis Model[1]

Factor analysis is a multivariate statistical technique which attempts to explain the correlations between a large set of observable random variables in terms of a minimal number of unobservable factors.

If a population of T time periods and N random variables are taken into consideration, where $N < T$, the factor analysis model initially assumes the existence of K underlying

1. The present section and the following three sections are largely based on Harman (1967) and Jöreskog and Sörbom (1979).

common factors, where $K < N$, and the following linear relationship :

$$\tilde{z}_{it} = b_{i1}\tilde{f}_{it} + b_{i2}\tilde{f}_{2t} + \ldots + b_{iK}\tilde{f}_{Kt} + \tilde{e}_{it} \tag{7.1}$$

where

$i = 1,2,\ldots,N$.

$t = 1,2,\ldots,T$.

$\tilde{z}_{it}$ = the value of the random variable i, for observation t.

$b_{i1},\ldots,b_{iK}$ = the <u>factor loadings</u> of the random variable i on the common factors $1,2,\ldots,K$ respectively.

$\tilde{f}_{it},\ldots,\tilde{f}_{Kt}$ = the values of the common factors $1,2,\ldots,K$ respectively for observation t. These values are called <u>factor scores</u>.

$\tilde{e}_{it}$ = the value of the specific factor to variable i, for observation t.

The factor analysis model is based upon the following assumptions :

(1) The distributions of $\tilde{f}_{1t},\tilde{f}_{2t},\ldots,\tilde{f}_{Kt}$ and $\tilde{e}_{it}$ are multivariate standarized normal and hence the joint distribution of the random variables $\tilde{z}_{1t},\tilde{z}_{2t},\ldots,\tilde{z}_{Nt}$ is multivariate standarized normal.

(2) The distributions of $\tilde{f}_{1t},\tilde{f}_{2t},\ldots,\tilde{f}_{Kt}$ and $\tilde{e}_{it}$ are stationary through time and thus the joint distrubtion of $\tilde{z}_{1t},\tilde{z}_{2t},\ldots,\tilde{z}_{Nt}$ is stationary through time.

(3) The covariance (correlation) matrix of the variables $\tilde{z}_{1t},\tilde{z}_{2t},\ldots,\tilde{z}_{Nt}$ is non-singular.

(4) The common factors are the same across various time periods.

(5) The factor loadings are stationary through time.

(6) The expected values on the common factors and the specific factors are equal to zero.

(7) The values of the common factors are independent of one another.

(8) The values of the specific factors are independent of one and another and of the values of the common factors.

From these assumptions it can be seen that there exist some additional assumptions to those required for the theoretical validity of the A.P.M. These additional assumptions are necessary conditions for the factor analysis procedure employed to test the A.P.M.

Furthermore the factor analysis model implies :

$$\sigma_i^2 = \frac{\sum_{t=1}^{T}(z_{it}z_{it})}{T} = \sum_{j=1}^{K} b_{ij}^2 + \sigma_{e_i}^2 \qquad (7.2)$$

where

$i = 1,2,\ldots,N$.

σ_i^2 = the variance of the variable $\tilde{z}_{it}$.

$\sigma_{e_i}^2$ = the variance of the specific factor $\tilde{e}_{it}$.

and

$$p_{ih} = \sum_{t=1}^{T} (z_{it} z_{ht}) = \sum_{j=1}^{K} b_{ij} b_{hj} \qquad (7.3)$$

where

$i,h = 1,2,\ldots,N,\ i \neq h$.

p_{ih} = the correlation between the variable z_{it} and z_{ht} .

The quantity $\sum_{j=1}^{K} b_{ij}^2$ is called the communality of the variable $\tilde{z}_{it}$ and it represents the portion of the total variance of the variable $\tilde{z}_{it}$ that is accounted for by the common factors. The variance $\sigma_{e_i}^2$ represents the portion of the total variance of the variable $\tilde{z}_{it}$ that is left unexplained by the common factors.

Next the quantity $\sum_{i=1}^{N} b_{ij}^2$ is the total amount of the variance in the population accounted for by the factor j, where $j = 1,2,\ldots,K$, while the quantity $\sum_{i=1}^{N} b_{ij}^2/N$ is the proportion of the population variance accounted for by the factor j, where $j = 1,2,\ldots,K$.

Lastly $\sum_{j=1}^{K} \sum_{i=1}^{N} b_{ij}^2/N$ is the proportion of the total population variance accounted for all the common factors $1,2,\ldots,K$.

If Σ and P are the covariance and correlation matrices of the variables $\tilde{z}_{1t},\tilde{z}_{2t},\ldots,\tilde{z}_{Nt}$, respectively then equations (7.2) and (7.3) can be rewritten in matrix notation as :

$$\Sigma = BB' + \Psi \tag{7.4}$$

and

$$P = BB' \tag{7.5}$$

where

B = the (NxK) matrix of factor loadings and B' is the transpose matrix of B

Ψ = the (NxN) diagonal matrix with $\sigma_{e_i}^2, i=1,2,\ldots,N$, along the diagonal.

Equation (7.5) indicates that the common factors explain the off-diagonal elements of P (namely the correlations) exactly.

In practice it is assumed that a random sample of T_1 observations, where $T_1 < T$, is drawn from the population of the T observations. The number of the random variables under consideration has to be less than T_1, because if the number of random variables exceeds T_1 then the resulting correlation matrix becomes singular.

If S and R are the unbiased estimates of the matrices Σ and P, respectively, then the factor analysis problem is to estimate the factor loadings matrix $\hat{\Lambda}$ and the diagonal matrix $\hat{\Psi}$ from S or R ; that is to estimate $\hat{\Lambda}$ and $\hat{\Psi}$ satisfying at least approximately the following equation :

$$S = \hat{\Lambda}\hat{\Lambda}' + \Psi \qquad (7.6)$$

or

$$R = \Lambda\hat{\Lambda}' \qquad (7.7)$$

7.2 Principal Component Analysis versus Common Factor Analysis

The main differences between the Principal Component Analysis (P.C.A.) and the Factor Analysis (F.A.) can be described as follows :

(1) The P.C.A.'s objective is to reproduce the total variance of a group random variables.
The F.A.'s objective is to reproduce the intercorrelations of a group of random variables.

(2) In P.C.A. all the components are needed to reproduce exactly the correlations among the variables. Few components, however, are required to explain a large portion of the total variance in the data.
In F.A., there exists by definition, a small number of factors that reproduce exactly the intercorrelations among the random variables. These factors, however, do not explain the same portion of variance as does the same number of principal

components.

(3) The P.C.A. does not assume a definite linear statistical model between the random variables and the common and specific factors.
The F.A. assumes a linear statistical model between the random variables and the common and specific factors.

(4) From the principal component analytic method the emerging components are linear combinations of the random variables.
From the factor analytic method the emerging factors are not linear combinations of the random variables.

(5) The P.C.A. does not require any assumption concerning the joint distribution of the random variables.
The F.A. assumes that the joint distribution among the random variables is multivariate normal.

(6) The P.C.A. assumes correlations among the specific factors of the random variables and it is concerned with the magnitudes of the specific factors.
The F.A. assumes zero correlations among the specific factors of the random variables and it is not concerned with the magnitudes of the specific factors.

(7) In P.C.A. the components are intercorrelated.
In F.A. there are no correlations among the factors.

(8) The principal component analytic method is not scale invariant .

There exist some methods between the factor analysis methods which are scale invariant (e.g. the maximum likelihood method, Rao's factor analytic method).

(9) In P.C.A. the components are unique in the sense that they stay the same as the number of the considered components is varied.
In F.A. the factor loadings are not unique in the sense that they change values as the number of factors changes.

(10) The P.C.A. can proceed if the covariance (correlation) matrix is singular.
The most factor analytic methods require a non-singular covariance (correlation) matrix.

7.3 Criticisms of the Factor Analytic Methods

The most important drawbacks of the factor analytic methods can be stated as follows :

(1) There exists a large number of assumptions behind the factor analysis model.

(2) The factor analytic techniques do not provide a method to specify the underlying factors. Consequently, the factor analytic methods may produce factors which represent only statistical artifacts.

(3) The number of underlying factors, say K, is unknown in

practice. In order to estimate the number of factors different values of K may be tried sequentially starting with K = 1. But different values of K give different factor loadings. That is the factor loadings are not unique. Even if it was possible to identify the underlying factors of an experimental design, then an additional problem would exist : the factor loadings change as the method of rotation changes.

In view of the drawbacks of the factor analytic methods Lawley and Maxwell (1971) noted :

> "It should always be kept firmly in mind that, except in artificial sampling experiments, the basic factor model is, like other models, useful only as an approximation to reality, and it should not be taken too seriously." (p.38).

The P.C.A. relies upon less restrictive assumptions from those of the F.A. Furthermore the P.C.A. is computationally simpler than the F.A. Thus a number of studies in the field of multivariate analysis concluded that the P.C.A. is preferable to the F.A.

It should be noted however, that the empirical examination of Ross' A.P.M. can be performed only by using common factor analytic methods. This is true since the A.P.M. assumes a security returns generating model in a similar fashion to the F.A. which assumes an underlying statistical linear model. Furthermore, a major assumption of the A.P.M.'s security returns generating model is the assumption regarding the independencies among the specific factors. This is a sufficient condition for the validity of the law of the large numbers, which in turn is needed to produce Ross' A.P.M. The F.A. also embodies the assumption concerning the independencies among the specific factors, while the P.C.A. does not require such an assumption.

Chen and Ingersoll (1982) proved that the A.P.M. can be an exact linear pricing ex-ante relationship if the assumption of the independence among the specific factors is relaxed. Even in this case the problem of identification of the components can not be avoided.

7.4 Factor Analytic Methods Used in Testing the Arbitrage Pricing Model

There are several factor analytic techniques available in the literature. The factor analytic techniques used in the A.P.T. are the following :

(1) The maximum likelihood factor analysis.

(2) Rao's or canonical factor analysis.

(3) The minres factor analysis.

(4) The alpha factor analysis.

The maximum likelihood method is usually preferable for the following reasons :

(i) It is scale invariant. That is the same interpretable factors and the same number of factors would be extracted regardless of the matrix scaling techniques applied to the data and regardless of whether the covariance or correlation matrix were used.

(ii) The estimating factor loadings are asymptotic efficient. That is they are asymptotically unbiased estimators having the smallest asymptotic variance among a group of asymptotically unbiased estimators.

(iii) The estimating factor loadings are consistent. That is they converge in probability to the population parameters being estimated as the sample size becomes larger.

(iv) There exists a chi-square test for the goodness of fit of the factor model.

The estimates obtained by Rao's factor analysis have been derived from principles other than the maximum likelihood principles. However, they satisfy the maximum likelihood equations and hence the previously mentioned statistical properties. Therefore they constitute another set of maximum likelihood estimates.

The minres factor analysis has less desirable properties than the maximum likelihood factor analysis. For example, it is not scale invariant and it only factor analyzes the correlation matrix. However, the solutions of the minres and maximum likelihood factor analysis are equivalent.

Lastly the solutions of the alpha and the maximum likelihood factor analysis are not equivalent. The main

difference between Rao's factor analysis and the alpha factor analysis can be described as follows :
Rao's factor analysis assumes that the given sample includes the entire randam variables and that the statistical inference about the estimation of the common factors of a population of observations is being made from a random sample of observations taken from the population.
The alpha factor analysis does not make statistical inference to a population of observations. Assuming that the variables are observed over a given population of observations, the alpha factor analysis is a method which enables us to make statistical inference about the estimation of the common factors of a population of variables from a random sample of variables taken from the population.

Roll and Ross (1980), Chen (1982), Reinganum (1981) Gibbons (1981) and Dhrymes, Friend and Gultekin (1982) adopted the maximum likelihood factor analysis because it has the previously mentioned statistical properties. Hughes (1982) utilized the minres factor analytic technique because its solution is equivalent to the maximum likelihood solution and its computational procedure is simpler than this of the maximum likelihood factor analysis.

Rao's factor analysis utilizes a statistical test for the goodness of fit of the factor model, while the alpha factor analysis uses the eigenvalue-one rule of thumb in testing the relevant number of factors. Therefore in order to compare the results derived by using a statistical test and the results derived by utilizing the rule of thumb Kryzanowski and Chau (1982) employed both Rao's and alpha factor analytic techniques.

Table 7.1 describes in summary the various factor models used in testing the A.P.T.

7.5 Conclusions

The P.C.A. and the F.A. have been used by a number of studies in portfolio theory, to investigate the structure of security (portfolio) returns. The P.C.A. and the F.A. are two similar approaches but they have different objectives. However, the superiority of the P.C.A. over the F.A. has been pointed out by a large number of studies.

The A.P.M. can be only tested by employing the techniques of the F.A. An effort of reformulating the theory of the A.P.M. and employing C.P.A. is useless since the identification problem of the components continues to exist.

Table 7.1 Factor analysis models used in the arbitrage pricing theory.

CHARACTERISTICS	MAXIMUM LIKELIHOOD FACTOR ANALYSIS	RAO FACTOR ANALYSIS	MINRES FACTOR ANALYSIS	ALPHA FACTOR ANALYSIS
Operating Principle	Maximizes the likelihood function of the correlation matrix of the entire set of random variables (securities)	Maximizes the square canonical correlation between a set of hypothesized factors for the population of observations(time periods) and the entire set of random variables (securities)	Minimizes the sum of the squared differencies between the off-diagonal elements of the original correlation matrix of the entire set of random variables (securities) and the reproduced correlation matrix.	Maximizes the square correlation between a set of hypothesized factors for the population of random variables(securities) and the set of factors found in a random sample which has been drawn from the population of variables.
Distribution Assumption	Multivariate normal	Multivariate normal	Multivariate normal	-
Assumption about an Underlying Statistical Model	Yes	Yes	Yes	Yes
Scale "Invariance"	Yes	Yes	No	No
Best Number of Factors	Statistical test	Statistical test	Statistical test	Eigenvalue greater than one.

CHAPTER 8

DATA SAMPLES AND METHODOLOGY

This chapter describes the methodology followed in this study. The first section states the specific research objectives, while the second, third and fourth sections are devoted to the nature and source of data, the sample period length and sample size, the selection criteria of the samples, the formation criteria of the groups and subgroups, respectively.
The fifth section presents the subperiods used in this study, whereas the rest of the chapter describes the statistical method employed in testing the desired assumptions.

8.1 Research Objectives

This study investigates empirically the validity of some important assumptions, necessary for unambiguous test of the A.P.M. using time series data from the London Stock Exchange (L.S.E.).

More specifically, it investigates the empirical validity of the following five assumptions :

(1) The distributions of security returns are normal ; the security mean returns and the covariance matrix of

security returns are intertemporally stationary.

(2) The number of factors affecting security returns is the same across various groups of different sizes.

(3) The number of factors which influence security returns is the same across various groups of the same size.

(4) The number of factors which affect security returns remains unchanged across various time periods for the same group of securities.

(5) The number of factors determining security returns remains unchanged across various time periods for various groups of securities.

8.2 Nature and Source of Data

The data used in this study was taken from the London Business School Share Price Database. Among other financial information the database contains monthly prices of the most ordinary shares (securities) that have traded in the London Stock Exchange (L.S.E.) from January 1955 to December 1981 (324 monthly observations).

From the London Share Price Database a file containing monthly log-returns on those securities was selected. The (nominal) monthly rate of return on a security i under continuous compounding was calculated as follows :

$$R_{it} = \ln \frac{P_{it} + D_{it}}{P_{it-1}}$$

where

R_{it} = the month t rate of return for security i assuming continuous compounding.

ln = the natural logarithm operator.

P_{it} = the last traded price for security i in month t.

D_{it} = the dividend for security i declared xd during month t adjusted to a month-end basis.

P_{it-1} = the last traded price of security i in month t-1 adjusted to the same base.

The logarithmic transformation is justified on two grounds :

(1) It approximates monthly price changes under continuous compounding.

(2) It improves the normality in the distribution of security returns since it reduces the skewness of the distribution.

8.3 Sample Period Length and Sample Size

Operationalization of the above mentioned tests requires the ulilization of time series data and the employment of multivariate statistical techniques. Consequently, the size of the sample and the sample period should be so that it fulfils the following requirements :

(i) Test Design Requirement : In order to test these assumptions a large sample, in terms of number of securities, and a large sample in terms of time periods are required. The second assumption is equivalent in testing the relationship between the number of factors and the group size . Such a relationship can be described clearly if one considers a large number of security groups having different sizes. Thus the larger the sample size the larger the number of security groups will be.
For the fourth and fifth assumptions a large number of observations is required so that to allow the entire set of observations to be divided into several different subperiods.

(ii) The Approximation of Ex-Ante Expectations with Ex-Post Measures Requirement : Since ex-ante models are derived and defined wholly with expectations, for testing purposes ex-post data is used in the place of ex-ante measures. Hence when the security market is efficient it is not unreasonable to consider ex-post returns as approximations to ex-ante returns. In an efficient market the difference between ex-post and ex-ante measures should average out to zero over reasonably long time lengths. So from this point of view the larger the time period the better will be the approximation among ex-post and ex-ante magnitudes.

(iii) Statistical Requirement : When a statistical hypothesis is tested the chief aim is to make a clear inference about an unknown population parameter (some unknown parameters of different populations) from a sample statistic (from statistics of different samples) that can be measured. For this reason it is necessary to take into consideration a large sample of securities and hence a high number of groups so that to increase the statistical power of the tests.
Moreover, it is well known that if the number of observations is sufficiently increased then the probability of finding linearly dependent security returns approaches to zero. Stated differently, if one increases sufficiently the number of observations then he reduces the multicollinearity between security returns. This is necessary since the multivariate statistical techniques which are used in this study require a non-singular correlation matrix.
Lastly when the correlation matrix of security returns is estimated using a large number of observations per security, then more efficient estimates of its entries are produced.

Comparing these three requirements it was initially decided to consider a large number of securities and a sample period comprised from a large number of observations. Unfortunately the data of the London Business School Share Price Database could not fulfil simultaneously all three requirements.

By examining different time periods it was found that the number of securities with continuous monthly data was decreased as the length of the time period was increased. Consequently the final decision was to take into account two different samples :

(1) A sample containing a large number of securities with observable returns over a relatively short time period.

(2) A sample containing a relatively small number of securities with observable returns over a long time period.

The consideration only of a sample containing a large number of securities with observable returns over a relatively short time period implies that it can be generated a large number of security groups and thus the statistical power of the tests will be increased. But the time period utilized in testing the intertemporal stationarity of the number of factors maybe produce multicollinearity problems. As a consequence the estimated covariance (correlation) matrix of security returns would be singular and hence the statistical techniques for testing the desired assumptions would be not applicable. The possible problems of multicollinearity may be solved by taking into consideration a sample containing securities with observable returns over a long time period. This sample will be comprised of a small number of securities and thus the statistical power of the tests concerning the relationship between the number of factors and the group size will be reduced.

However, by considering two different samples the disadvantages affecting the statistical tests of this study can be greatly minimized.

8.4 Selection of the Time Period and Selection Criteria of the Samples

The total firms which quoted in the L.S.E. since January 1955 are more than 3,500. The first sample period was selected to satisfy two objectives :

(i) To produce a large number of securities.

(ii) To contain enough observations so that to allow the entire sample period to be divided into two non-overlapping periods.

In view of these two objectives it was decided to choose a 10 year period, 120 monthly observations for each security. The first sample period begins on January 1st. 1972 and it ends on December 31st. 1981.

Initially a firm to qualify for inclusion in the first sample had to satisfy the following criterion :

(1) To be listed continuously on the L.S.E. during the 10 year period.

From the whole number of the firms only 899 firms had a continuous data during the 10 year period. The 899 firms constitute about 26 per cent of the total number of firms quoted in the L.S.E. since 1955.

The first sample criterion probably introduces a survival bias in the sense that it has only included firms in existence during the 10 year period. This is, however, a natural consequence of all studies of this type, working with a time series data which has a fixed length.

Unfortunately among those firms there were firms whose securities had at least one month with no recorded trade. Dimson (1979) pointed out that such non-trading securities have high auto-correlation of returns. Auto-correlated security returns imply fewer degrees of freedom in the sample and hence biased estimated variances of security returns. The biased security variances will produce a biased covariance (correlation) matrix of security returns. The tests of this study utilize the covariance (correlation) matrix of security returns. Thus, if a biased covariance (correlation) matrix is used then the producing results will be biased.

Consequently it was necessary to consider a further criterion for inclusion of a firm in the sample. That is :

(2) Over the entire sample period of 120 monthly observations, securities having in more than three months no recorded trade are excluded.

It was decided to include securities having in two or three months no recorded trade over the 10 year period, because there were only few securities having in two or

three months no recorded trade and such months were not appeared suquentially.
From the 899 firms only 672 firms satisfied the second criterion.[1] The last chosen number of firms constitutes about 19 per cent of the whole population of securities quoted in the L.S.E. since 1955.

In this case the new sample may be biased by the deletion of the securities which had in more than three months no recorded trade. However, such a bias may be infinitesimally small.

The first sample utilized in this study is described in Table 8.1.

The second sample period was selected to satisfy the following two goals :

(i) To increase as much as possible the number of observations than the previous sample period.

(ii) To produce a reasonable number of securities with continuous monthly data for the entire sample period.

---

1. The London Business School Share Price Database contains for each monthly return an equivalent non-trading indicator. This basically indicates the number of days before the end of the month that the last trade occurred. The number 32 associated with a particular month indicates that such a share was not traded this month.

Table 8.1 Description of sample A.

Source:	London Share Price Database Graduate Business School University of London Monthly Returns File
Sample Period:	January 1972 - December 1981
Sample Size per Security:	120 monthly observations
Selection Criteria:	(i) Securities must be continuously listed on the London Stock Exchange for the entire sample period. (ii) Securities must have no less than 117 trading months out of the 120 months.
Number of Selected Securities:	672
Basic Data Unit:	Log-returns adjusted to a month-end basis.

With the aid of these two goals it was selected the sample period from November 1st. 1956 to December 31st. 1981, that is 302 monthly observations from each security.

The securities which have continuous monthly data during the second sample period are 200 only. In this case it is only considered about 6 per cent of the entire number of securities quoted in the L.S.E. since 1955. By considering 200 securities the number of securities of the first sample was decreased by 30 per cent. Moreover it is notified that the results which will be presented in the following chapters, by using the second sample probably contain a survival bias.
The second criterion which imposed upon the first sample was not necessary to be imposed upon the second sample, because almost all the firms of the second sample, are large firms having frequently traded securities.

Table 8.2 describes the second sample used in this study.

8.4.1 The Groups

The number of the securities of each sample was initially divided into several random master groups of equal size. Before deciding the size of the master group the following were taken into consideration :

Table 8.2 Description of sample B .

Source:	London Share Price Database Graduate Business School University of London Monthly Returns File
Sample Period:	November 1956- December 1981
Sample Size per Security:	302 monthly returns
Selection Criterion:	Securities must be continuously listed on the London Stock Exchange for the entire sample period.
Number of Selected Securities:	200
Basic Data Unit:	Log-returns adjusted to a month-end basis.

(i) It is necessary to formulate a substantial number of master groups to provide grounds for statistical inference of the designed tests.

(ii) In order to increase the statistical power of the test concerning the relationship between the number of factors and the group size it is necessary to generate from each master group a considerable number of subgroups.

(iii) The half size of the sample in terms of time periods has to exceed the size of the master groups. If the number of variables is greater than the number of observations then the resulting covariance (correlation) matrix is singular. But the empirical examination of the stated assumptions requires non-singular covariance (correlation) matrix.

Having in mind these three requirements it was decided the size of the master groups of the first sample to be equal to 42. Such a size produces enough master groups and each master group can generate enough subgroups. Lastly, 42 is smaller than 120/2 and hence one of the cases to derive a singular covariance (correlation) matrix is excluded.

Similarly for the second sample it was decided the size of the master groups to be equal to 40.

The 672 (200) company numbers were listed in ascending order and 16 (5) groups each consisting of 42 (40) securities were drawn. By forming the master groups following this procedure a randomly selection of the members of each group is achieved.

From each master group of the first (second) sample were formed 7 subgroups containing 5, 10, 15, 21, 26, 31 and 36 (5, 10, 15, 20, 25, 30 and 35) securities, respectively. Each subgroup contains the first ϑ (ϑ_1), where ϑ = 5, 10, 15, 21, 31, 36 (ϑ_1 = 5, 10, 15, 20, 25, 30, 35), securities of the corresponding master group.

8.5 The subperiods

The subperiods in each sample were chosen to satisfy the following criteria :

(1) To be non-overlapping. This requirement is extremely important for our analysis. Indeed by utilizing non-overlapping subperiods,the possibility to derive replicable factors having influence on security returns during the time period covered by a sommon length of the subperiods is excluded.

(2) To have equal lengths so that the performed tests are equally affected by the sample size in terms of time periods.

(3) To contain a number of return observations that is greater than the size of the master group so that to exclude one of the possibilities to derive a

singular covariance (correlation) matrix.

According to these requirements the entire period of the 120 monthly observations was divided into two non-overlapping subperiods of 60 observations each. On the other hand the entire period of the 302 observations was divided into two different sets of subperiods. The first set is made from three non-overlapping subperiods of 100 observations each, while the second set is generated from two non-overlapping subperiods of 151 observations each.

Table 8.3 describes the chronological division of each sample.

8.6 Statistical Tests

Since for each assumption is necessary to perform more than one statistical test, Table 8.4 gives a summary of the various tests.

Below it is noted some general remarks concerning the tests of Table 8.4 (see page 184).
The test of normality is necessary because the tests regarding the intertemporal stationarity of the covariance or correlation matrix are very sensitive to departures from normality. The second, the third and the fourth tests are concerned with the intertemporal stationarity of the security mean returns. However, if one considers samples

Table 8.3 The subperiods .

	TIME INTERVAL COVERED BY THE SUBPERIODS	NUMBER OF OBSERVATIONS PER SECURITY
SAMPLE A SUBPERIOD: 1/1972-12/1981 NUMBER OF SECURITIES:672	1/1972-12/1976	60
	1/1977-12/1981	60
SAMPLE B SUBPERIOD: 11/1956-12/1981 NUMBER OF SECURITIES:200	11/1956-2/1965	100
	3/1965-6/1973	100
	7/1973-12/1981	100
	11/1956-5/1969	151
	6/1969-12/1981	151

Table 8.4 Assumptions and statistical tests.

ASSUMPTION	STATISTICAL TESTS
(1) The security mean returns, the covariance matrix of security returns and the correlation matrix of security returns are intertemporally stationary.	(i) A test for normality. (ii) A test of the homogeneity of two security variances. (iii) A test for the difference between two security means when the populations have common variances. (iv) A test of the difference between two security means when the populations have unequal variances. (v) A test of the homogeneity of two covariance matrices of security returns. (vi) A test of the homogeneity of two correlation matrices of security returns.
(2) The number of factors affecting security returns is the same across various groups of different sizes. (3) The number of factors which influence security returns is the same across various groups of the same size.	(vii) A test for a complete independence of the correlation matrix of security returns. (viii) A test for the goodness of fit of the factor model across different groups of various sizes and across different groups of the same size.
(4) The number of factors which affect security returns remains unchanged across various time periods for the same group of securities (5) The number of factors affecting security returns remains unchanged across various time periods for various groups of securities of different sizes.	(ix) A test for a complete independence of the correlation matrix of security returns. (x) A test for the goodness of fit of the factor model across various time periods for the same group of securities and for different groups of securities.

of equal sizes in terms of time periods then the second test is unimportant, because the third and fourth tests produce equal testing statistics (more details are given in Section 8.10).

Finally the sixth test will be performed if the covariance matrix security returns is not intertemporally stationary. In this case if the correlation matrix is intertemporally stationary such a matrix will be used in the remaining tests.

8.7 First Test : A Test for Normality

Assumption of the Test[1]

For each security a set of observations represents a sample of independent observations in the sense that the occurrence (or nonoccurrence) of one observation of the set cannot influence (or be influenced by) the occurrence of another observation of the set.

Description of the Test

This test examines whether the monthly observations of security returns are drawn from a normal population of observations.
In this case the null and the alternative hypothesis may be expressed in the following form :

---

1. For a detailed description of this test, see David, Hartley and Pearson (1954) and Fama (1976).

$$H_o : \tilde{R}_i \approx N(\mu, \sigma^2)$$

versus

$$H_1 : \tilde{R}_i \not\approx N(\mu, \sigma^2)$$

where

$\tilde{R}_i$ = the random return on a security i.

$N(\mu, 0^2)$ = a symbol indicating that the random variable $\tilde{R}_i$ is normally distributed with mean μ and variance σ^2 .

The test statistic is shown in equation (8.1) below:

$$SR = \frac{\max(R_i) - \min(R_i)}{s_{R_i}} \tag{8.1}$$

where

SR = studentized range.

$\max(R_i)$ = the maximum value of the secirity i's monthly return over the sample period.

$\min(R_i)$ = the minimum value of the security i's monthly return over the sample period.

s_{R_i} = the estimated standard deviation of the return of the i^{th} security.

The null hypothesis that the security returns can be considered as drawings from a normal disrtibution is accepted if :

$$SR < SR\ (f, T_1) \tag{8.2}$$

where

$SR(f, T_1)$ = the fractile of the distrubtion of the studentized range in samples of size T_1, with T_1 denoting the total number of return observations. By definition, a fractile, is a value below which a speci fied fraction of the data must lie.

8.8 Second Test : Test for the Homogeneity of two Variances

Assumptions of the Test[1]

(1) For each security can be generated two equally size random samples of return observations, drawn from two normally distributed populations with unknown means variances, μ_i, μ_i^*, σ_i^2, σ_i^{*2}, respectively.

(2) For each security the two random samples of return observations are independent.

Description of the Test

This test determines whether or not the variance of the return on a security is intertemporally stationary.

---

1. The statistical theory underlying this test is given in Blalock (1972).

Thus the following hypothesis is tested :

$$H_o : \sigma_i^2 = \sigma_i^{*2}$$

against

$$H_1 : \sigma_i^2 \neq \sigma_i^{*2}$$

The test statistic is given by :

$$F_1 = \frac{s_i^2}{s_i^{*2}} \qquad (8.3)$$

where

$s_i^2 > s_i^{*2}$

F_1 has an F-distribution with (T_2-1, T_2-1) degrees of freedom.

T_2 = the size of each random sample in terms of time periods.

s_i^2 = the variance of returns on the security i, estimated using the first T_2 observations.

s_i^{*2} = the variance of returns on the security i, estimated using the second T_2 observations.

The null hypothesis for testing $H_o : \sigma_i^2 = \sigma_i^{*2}$ is accepted if :

$$F_1 < F_{a,T_2-1,T_2-1} \qquad (8.4)$$

where

F_{a,T_2-1,T_2-1} = a critical value of the test statistic.

a = a level of significance.

8.9 Third Test : Test for the Difference between two Means when the Populations have Common Variance

Assumptions of the Test[1]

(1) For each security can be formed two equally sized random samples of return observations, drawn from two normally distributed populations with unknown means and variances μ_i, μ_i^*, σ_i^2, σ_i^{*2}, respectively.

(2) The unknown populations variances are equal, i.e. $\sigma_i^2 = \sigma_i^{*2} = \dot{\sigma}_i^2$, for each i.

(3) For each security the two random samples of return observations are independent.

Description of the Test

This test examines the intertemporal stationarity of a security's mean when the security's variance of return is intertemporally stationary.

The null and the alternative hypotheses are set up as follows :

---

1. A detailed description of this test can be found in Blalock (1972).

$$H_o: \mu_i = \mu_i^*$$

versus

$$H_1: \mu_i \neq \mu_i^*$$

The following equation represents the test statistic:

$$t_c = \frac{r_i - r_i^*}{\dot{s}_i\sqrt{\frac{2}{T_2}}} \tag{8.5}$$

where

t_c has a t-distribution with $2(T_2-1)$ degrees of freedom.

T_2= the size of each random sample in terms of time periods.

r_i= the mean return onthe security i,estimated by utilizing the first T_2 observations.

r_i^*= the mean return on the security i,estimated by utilizing the second T_2 observations.

$\dot{s}_i=\sqrt{\frac{s_i^2 + s_i^{*2}}{2}}$ = a best and unbiased estimator of $\dot{\sigma}_i$.

The null hypothesis that there is no difference between μ_i and μ_i^* is accepted if :

$$|t_c| < t_{a,2(T_2-1)} \tag{8.6}$$

where

a= a level of significance.

8.10 Fourth Test : Test for the Difference between two Means when the Populations have Unequal Variances.

Assumptions of the Test[1]

(1) For each security can be generated two equally size random samples of return observations, drawn from two normally distributed populations with unknown means and variances μ_i, μ_i^*, σ_i^2, σ_i^{*2}, respectively .

(2) The unknown populations variances are not equal.

(3) For each security the two random samples of return observations are independent.

Description of the Test

This is test is concerned with the examination of the intertemporal stationarity of a security's mean when the security's variance of return is not intertemporal stationary. In this case the null and the alternative hypotheses are the same as in the previous case. That is :

$$H_o: \mu_i = \mu_i^*$$

against

$$H_1: \mu_i \neq \mu_i^*$$

Since the populations have unequal variances the statistic t_c cannot be computed as it was computed in the previous case. Instead the following approximation of t_c is considered :

---

1. The details of this test are presented in Blalock (1972).

$$t_{c1} = \frac{r_i - r_i^*}{\sqrt{\frac{s_i^2 + s_i^{*2}}{T_2}}} \qquad (8.7)$$

In such a case the test statistic t_c, will be approximately distributed as a t-distribution with degrees of freedom given by :

$$df = \frac{(s_i^2 + s_i^{*2})(T_2 - 1)}{(s_i^2)^2 + (s_i^{*2})^2} \qquad (8.8)$$

It is evident that the t-statistic given by equation (8.5) and the t-statistic (8.7) are the same. This is implied because it was assumed samples having equal numbers of observations. In this case, the assumption of equal populations variances is relatively unimportant. However, the second test was performed in order to attain an idea concerning the intertemporal stationarity of the security variances.[1]

8.11 Fifth Test : A Test for the Homogeneity of two Covariance Matrices

Assumptions of the Test[2]

(1) For each security can be formed two equally random samples of return observations, drawn from two

1. The computer programmes used in the second, third and fourth tests are contained in the Statistical Package for the Social Sciences (S.P.S.S.).

2. This statistical test is described in Morrison (1967).

multivariate normally distributed populations with a known cavariance matrices $\sum_g$, and $\sum_g^*$, respectively.

(2) For each security the two random samples of return observations are independent.

Description of the Test

This test deals with the problem of investigating the intertemporal stationarity of the covariance matrix. For this test the appropriate hypotheses would be :

$$H_o : \sum_g = \sum_g^*$$

versus

$$H_1 : \sum_g \neq \sum_g^*$$

The test statistic is described by the following equation :

$$C = \left[T_2 - 1 - \frac{(2p^2 + 3p - 1)}{4(p + 1)} \right] \ln \frac{\frac{\left| S_g + S_g^* \right|^2}{4}}{\left| S_g \right| \left| S_g^* \right|} \qquad (8.9)$$

where

C is approximately distributed as a chi-squared variate with p(p+1)/2 degrees of freedom.

p=the number of securities in the group.

T_2=the size of each random sample in terms of time periods.

ln =the natural logarithm operator.

S_g = an unbiased estimate of the population covariance matrix $\sum_g$.

S_g^* = an unbiased estimate of the population covariance matrix $\sum_g^*$.

The null hypothesis of the homogeneity of the covariance matrices $\sum_g$ and $\sum_g^*$ is accepted if :

$$C < X^2_{a,\frac{1}{2}p(p+1)} \qquad (8.10)$$

where

a = a level of significance[1].

8.12 Sixth Test : A Test for the Homogeneity of two Correlation Matrices

Assumptions of the Test[2]

(1) For each security can be chosen two equally size samples which can be regarded as randomly drawings from two multivariate normally distributed populations with unknown correlation matrices P_g and P_g^*,respectively.

(2) For each security the two random samples of return observations are independent.

Description of the Test

This test provides a means of examining the intertemporal stationarity of the correlation matrix. Here the null and the hypotheses are stated as follows :

---

1. The computer programme for this test was developed by G. Thanassoulas.
2. This test was developed by Jennrich (1970).

$$H_o : P_g = P_g^*$$

against

$$H_1 : P_g \neq P_g^*$$

The test statistic can be expressed as :

$$C_1 = \tfrac{1}{2}\mathrm{tr}(Z^2) - dg'(Z)S^{-1}dg(Z) \tag{8.11}$$

where

C_1 is distributed as a chi-squared variate with $p(p-1)/2$ degrees of freedom.

p = the number of securities in the group.

$$Z = \sqrt{\frac{T_1}{2}}\left[\frac{R_g + R_g^*}{2}\right]^{-1}(R_g - R_g^*) .$$

T_1 = the size of each random sample in terms of time periods.

R_g = an unbiased estimate of the population correlation matrix P_g.

R_g^* = an unbiased estimate of the population correlation matrix P_g^*.

$Z^2 = ZZ$.

$\mathrm{tr}(.)$ = the trace of the square matrix Z^2.

$S = I + r_{ij}r^{ij}$.

I = an identity matrix.

r_{ij} = the entry of the matrix $\dfrac{R_g + R_g^*}{2}$ appearing in the i^{th} row and the J^{th} column.

r^{ij} = the entry of the matrix $\left[\dfrac{R_g + R_g^*}{2}\right]^{-1}$ appearing in the i^{th} row and the j^{th} column .

$dg(Z)$ = the vector whose entries are the diagonal elements of the matrix Z.

$dg'(Z)$ = the transpose of $dg(Z)$.

The null hypothesis concerning the homogeneity of the correlation matrices P_g and P_g is accepted if :

$$C_1 < X^2_{a,\frac{1}{2}p(p-1)} \qquad (8.12)$$

where

a = a level of significance.[1]

8.13 Seventh Test : A Test for the Complete Independence of the Correlation Matrix

Assumptions of the Test[2]

(1) For each security it can be drawn a random sample from a multivariate normally distributed population with a known correlation matrix P_g.

(2) The correlation matrix is non-singular.

---

1. The computer programme for this test was developed by G. Thanassoulas.

2. This test was suggested by Bartlett (1950).

Description of the Test

This test examines the suitability of the correlation matrix for factor analysis. Here the null and the alternative hypotheses are presented as follows :

$$H_o : P_g = I$$

versus

$$H_1 : P_g \neq I$$

where

I = an identity matrix,

The test statistic has the following form :

$$C_2 = -(T_1 - 1 - \frac{2p+5}{6})\ln|R_g| \tag{8.13}$$

where

C_2 is approximately distributed as a chi-squared variate with $\frac{1}{2}p\,(p-1)$ degrees of freedom.

p = the number of securities in the group.

T_1 = the size of the random sample in terms of time periods.

ln = the natural logarithm operator.

R_g = an unbiased estimate of the population correlation matrix P_g.

The null hypothesis of the complete independence is accepted if :

$$C_2 < X^2_{a,\frac{1}{2}p(p-1)} \tag{8.14}$$

where

a = a level of significant.

8.14 Eighth Test : Test for the Goodness of Fit of the Factor Model across Different Groups of Various Sizes and across Different Groups of the Same Size

It was explained in the previous chapter the superiority of the maximum likelihood factor analytic method versus other factor analytic methods. It was also pointed out that the estimates derived from Rao's factor analysis constitute another set of maximum likelihood estimates.

Furthermore, the algorithm developed by Jörestog to solve the maximum likelihood estimation equations is extremely sensitive to the ill-condition of the correlation matrix (i.e. to a correlation matrix that has a very small determinant). When the correlation matrix of security returns has a very small determinant the maximum likelihood factor analysis cannot be done.

For these reasons it was decided to employ in this study the factor analyitic method of Rao.

8.14.1 Rao's Factor Analytic Method

The operating principle of Rao's factoring methods is the estimation of a factor solution which maximizes the squared canonical correlation between a set of hypothesized factors and the set of variables (securities).

Rao's factor analysis model fits the description of the general factor analysis model described in Chapter 7. Rao's model assumes that the number of common factors is specified in advance to be equal K, where K is an integer number, $K < N$ and N is the number of securities under consideration.

Under Rao's factoring method the first order conditions for the maximum value of the squared canonical correlation between the set of hypothesized factors and the set of variables (securities) can be written as :

$$|BB' - a^2R| = 0 \qquad (8.15)$$

where

B = the (N x K) matrix of factor loadings and B' denotes the transpose of B.

R = the (N x N) estimated correlation matrix of security returns.

a^2 = the square of the canonical correlation coefficient between the set of hypothesized factors and the set of variables.

Since Rao's factor analysis is scale invariant the estimation equations of the factor loadings can be expressed in terms of the correlations rather the covariances. That is :

$$R = BB' + \Psi \tag{8.16}$$

where

Ψ = the (N x N) estimated diagonal matrix with the variances of the specific factors along the diagonal.

Substituting equation (8.16) into equation (8.15) and rearranging terms one produces :

$$|R - \lambda\Psi| = 0 \tag{8.17}$$

where

$$\lambda = \frac{1}{1 - a^2}\text{, with } a^2 \neq 1 .$$

Since is a diagonal matrix it is possible to write

$$\Psi^{-1} = \Psi^{-\frac{1}{2}}\Psi^{-\frac{1}{2}} \tag{8.18}$$

With the aid of equation (8.18) equation (8.17) can be expressed as :

$$|\Psi^{-\frac{1}{2}}R\,\Psi^{-\frac{1}{2}} - \lambda I| = 0 \tag{8.19}$$

where

I = the $(N \times N)$ identity matrix.

Equation (8.19) indicates that the desired eigenvalues λ are the eigenvalues of the $(N \times N)$ symmetric matrix $\Psi^{-\frac{1}{2}} R \, \Psi^{-\frac{1}{2}}$.
The correlations between a security's return and the values on the common factors are functions of the first K eigenvectors, where $K < N$, and each correlation is equal to the factor loadings to the security returns on the corresponding factor. Consequently, one can express the diagonal element of $\Psi^{-\frac{1}{2}}$ as

$$g_i = \sqrt{(\lambda_1 - 1)a_{1i}^2 + (\lambda_2 - 1)a_{2i}^2 + \ldots + (\lambda_K - 1)a_{Ki}^2 + 1} \quad (8.20)$$

where

$i = 1, 2, \ldots, K, K+1, \ldots, N$.

$\lambda_1, \lambda_2, \ldots, \lambda_K$ = the first K largest eigenvalues of equation (8.18).

$A_m = (a_{m1}, a_{m2}, \ldots, a_{mN})$ = the eigenvector corresponding to the eigenvalue λ_m, $m = 1.2, \ldots, K$.

Equation (8.19) can be rewritten as

$$\left| GRG - \lambda I \right| = 0 \quad (8.21)$$

where

$G = \Psi^{-\frac{1}{2}}$

Rao pointed out that a better approximation of the diagonal elements of G is given by the following formula:

$$g_{i1}=\sqrt{(\frac{\lambda_1}{\lambda_o}-1)a_{1i}^2+(\frac{\lambda_2}{\lambda_o}-1)a_{1i}^2+\ldots+(\frac{\lambda_K}{\lambda_o}-1)a_{Ki}^2+1} \quad (8.22)$$

with

$$\lambda_o=\frac{\sum_{i=1}^{N} g_i^2-\lambda_1-\lambda_2-\ldots-\lambda_K}{N-K} \quad (8.23)$$

where g_i is given by equation (8.20) .

In practice the diagonal elements of G and the factor loadings are both unknown. Thus their computation requires the employment of an iterative process.

Such a process contains the following steps :

(1) Compute the (N x N) correlation matrix of security returns and its inverse.

(2) Calculate the N eigenvalues of the correlation matrix and the corresponding eigenvectors to those eigenvalues.

(3) Estimate the diagonal entries of the matrix G by utilizing equation (8.22).

(4) Compute the N eigenvalues of the symmetric matrix GRG.

(5) Calculate a new diaganol matrix, call it G_1, by using equation (8.22), and the first K largest eigenvalues and their corresponding eigenvectors obtained from the previous iteration (see step 4), where $K < N$.

(6) Estimate the N eigenvalues of the symmetric matrix $G_1 R G_1$.

(7) Compute a new diagonal matrix, call it G_2, by employing equation (8.22), and the first K largest eigenvalues and the corresponding eigenvectors obtained from the previous iteration (see step 6), where $K < N$.

(8) Estimate the N eigenvalues of the symmetric matrix $G_2 R G_2$.

(9) Repeat the iteration process until it is found that the maximum values of two successive estimates, computed with the aid of equation (8.22) , do not defer by more than some predetermined real number.

(10) Compute the factor loadings by making use of the following equation :

$$B_\xi = G_\xi^{-1} A_\xi \sqrt{\Lambda_\xi - i}$$

where

B_ξ = the (N x K) matrix of factor loadings.

G_ξ = the (N x N) diagonal matrix estimated from the final iteration.

Λ_ξ = the (N x 1) column vector with entries the eigenvalues of the symmetric matrix G RG .

i = the (N x 1) unit vector. The square root is referred to each entry of the vector $\Lambda_\xi - i$.

A_ξ = the (K x N) matrix with rows the eigenvectors of the symmetric matrix G RG .

(11) Estimate the final communalities by employing the formula :

$$C_{\xi i} = \sum_{m=1}^{K} b_{im}^2$$

where

$i = 1,2,\ldots,N$.

b_{im} = the m-entry of the i^{th} row of the matrix B_{ξ} .

8.14.2 A Test for the Goodness of Fit of Rao's Factor Model

One of the main advantages of Rao's factor analysis is that it provides a test for the relevant number of factors. Indeed using Rao's factor analytic technique the following hypothesis is tested :

$$H_o : R_g = BB'$$

versus

$$H_1 : R_g \neq BB'$$

The test statistic is described by the following equation :

$$C_3 = (T_1 - 1 - \frac{2N+5}{6} - \frac{2}{3}K)\left[\ln(\lambda_{K+1}\cdots\lambda_N) - (N-K)\ln(\frac{\lambda_{K+1}+\cdots+\lambda_N}{N-K})\right] \tag{8.24}$$

where

C_3 is distributed as a chi-squared variate with

$\frac{1}{2}\left[(N-K)^2 - N - K\right]$ degress of freedom.

T_1 = the size of the random samples in terms of time periods.

ln = the natural logarithm operator.

N = the number of securities.

K = a given number of factors.

$\lambda_{K+1}, \ldots, \lambda_N$ = the least (N - K) characteristic roots of equation (8.21) at any stage of iteration.

The null hypothesis that there exist exactly K factors is accepted if :

$$C_3 < X^2_{a,\frac{1}{2}} \quad (N - K)^2 - N - K \qquad (8.25)$$

where

a = a level of significance.

The idea behind this test is that by "removing" the first K largest characteristic roots of the square matrix GRG the equality of the remaining N - K roots can be examined. If the remaining N - K roots of the matrix GRG are equal, then there is no point in trying to extract any more factors.[1]

---

1 For this test it was used a computer contained in the S.P.S.S.

8.15 Conclusions

This research utilizes two different samples. The first sample contains 672 securities with no missing observations for the period January 1972 - December 1981. The second sample is comprised of 200 securities for which there was a complete history of monthly return data from November 1956 - December 1981.

This study employs 10 statistical tests in order to examine whether the A.P.M. can be tested unambiguously by using a time series data taken from the L.S.E.

CHAPTER 9

THE DISTRIBUTION OF SECURITY RETURNS : SOME EMPIRICAL EVIDENCE.

Empirical validation of the A.P.M. is based on the premise that security returns are drawn from normal distributions and these distributions are stationary through time. This chapter examines the assumption of the normal distribution of security monthly returns and investigates the assumption regarding the intertemporal stationarity of the distributions of security returns.

These assumptions are verified empirically by utilizing the following statistical tests, respectively :

(1) The studentized range test for normality.

(2) A t-square test for the intertemporal stationarity of security mean returns and a chi-square test for the intertemporal stationarity of the covariance (correlation) matrix of security returns.

Before presenting the empirical results of the preceding tests the characteristics of the joint distribution of security returns are estimated.

Table 9.1 and 9.2 summarize the characteristics of the joint distribution of returns on securities,

Table 9.1 Average values of the first two moments and quintiles of the security returns distributions.
Sample A: Period : 1/1972-12/1981, Number of securities: 672 .

NUMBER OF OBSERVATIONS	MEAN	VARIANCE	STANDARD DEVIATION	LOW	QUINTILES				HIGH
					.20	.40	.60	.80	
120	.008	.000147	.11	-.357	-.058	-.006	.071	.094	.0464

Table 9.2 Average values of the first two moments and quintiles of the security returns distributions.
Sample B : Period: 11/1956-12/1981, Number of securities: 200 .

NUMBER OF OBSERVATIONS	MEAN	VARIANCE	STANDARD DEVIATION	LOW	QUINTILES				HIGH
					.20	.40	.60	.80	
302	.0105	.00001	.091	-.0348	-.045	.07	.062	.078	.456

using samples A and B, respectively.

Table 9.1 shows that utilizing sample A, the average of the security means over the first sample period is .008, the average of the security variances is .00014 and the average of the security standard deviations is .11 .

The results of Table 9.2 reveal that using sample B, the average of the security means over the second sample period is .001, the average of the security variances is .0001 and the average of the security standard deviations is .91 .

9.1 Statistical Testing Procedure for the Studentized Range Test and the Empirical Results

The studentized range test for normality is separately adopted for each of the two samples (A and B) considered in this work. Such a test requires the following three-step procedure.

(1) For each security in the sample the range and the standard deviation of its distribution are calculated.

(2) For each security in the sample the studentized range given by equation (8.1) is computed.

(3) Each value of the studentized range is compared with the .995 fractile of the distribution of the studenized range in sample of size T_1 (the total number of return

observations per security) and a decision concerning the validity of the normality assumption is made.

Table 9.3 summarizes the results of the studentized range test for sample A, using two nonoverlapping subperiods of 60 monthly observations each.

The empirical findings of the studentized range test for sample B using three nonoverlapping subperiods of 100 monthly observations each and two nonoverlapping subperiods of 151 monthly observations each are summarized in Table 10.4 .

The results shown in Table 9.3 indicate that when one uses the subperiod 1/1972 - 12/1976 (1/1977 - 12/1981) 71 (86.1) per cent of the securities did not exceed the value (6.09) of the .995 fractile of the distribution of the studentized range in samples of 60 observations from a normal distribution. Therefore these results support that for the subperiods of sample A the monthly security returns are close enough to normal. Similarly from Table 9.4 can be inferred that the security monthly returns are approximately normal. Such an approximation is much closer to normal when the subperiods 11/1956 -2/1965, 3/1965 - 6/1973 and 11/1956 - 5/ 1969 are examined. The different results for the subperiods 7/1973 - 10/1981 and 6/1969 - 12/1981 can be attributed to the abnormal market behaviour during the 1974 - 1975 period. During these subperiods there exist more leptokuitic distributions

Table 9.3 A test for normality using the studentized range in samples comprised of 60 monthly observations.
Sample A : Period 1/1972-12/1981, Number of securities :672.

SUBPERIOD	SAMPLE SIZE (MONTHLY OBSERVATIONS)	PERCENTAGE OF SECURITIES WITH NORMAL DISTRIBUTED RATES OF RETURN
VALUE OF FRACTILE AT .995 : 6.09		
1/1972-12/1976	60	71
1/1977-12/1981	60	86.1

Table 9.4 A test for normality using the studentized range in samples comprised of 100 and 151 monthly observations.
Sample B : Period : 11/1956-12/1981, Number of securities:200 .

SUBPERIOD :	SAMPLE SIZE (MONTHLY OBSERVATIONS)	PERCENTAGE OF SECUTITIES WITH NORMAL DISRTIBUTED RATES OF RETURN
VALUE OF FRACTILE AT .995:6.09		
11/1956-2/1965	100	80
3/1965-6/1973	100	88
7/1973-10/1981	100	59
11/1956-5/1969	151	75
6/1969-12/1981	151	56

of returns on securities compared with the distributions of other subperiods.

The approximately normal distributions of security returns imply that it is safe to apply the common statistical tests, especially those which are sensitive to departures from normality.

Brealy (1979), however, reported evidence shown that the distributions of the daily U.K. security returns are non-normal. His conclusions are different to those presented in this work. This is probably due to the fact that daily returns fluctuate less than monthly returns so that the distributions of daily returns are more peaked about their means and the relative frequencies of the extreme daily returns are greater than those of the monthly returns. As a consequence daily security returns are more leptokuitic relative to monthly security returns.

9.2 Statistical Testing Procedure for the Intertemporal Stationary Distributions of Security Returns and the Empirical Results

To test the intertemporal stationarity of the distributions of security returns tests 2 to 6 described in Table 8.6 are used ; that is, a test of the homogeneity of two security variances, a test for the difference between two security means when the populations have common (unequal) variances and a test for the homogeneity of two covariance (correlation) matrices of security returns. The above tests are operationalized separately for each of the two samples (A and B) under consideration in this study.

Sample A of 672 securities is broken down into sixteen random master groups of size 42 and from each group are drawn two subgroups containing 21 and 10 securities respectively. The first sample period of 120 months is divided into two nonoverlapping subperiods. These subperiods are from January 1972 until December 1976 and from January 1977 until December 1981.

The sample B consisting of 200 securities is broken down into five random master groups of size 40 and from each master group are drawn two subgroups containing 20 and 10 securities respectively. From the second sample period of the 302 months six subperiods are generated. The subperiods are from November 1956 until January 1963, February 1963 until April 1969, May 1969 until July 1975, August 1975 until October 1981, November 1956 until May 1969 and June 1969 until December 1981.

Homogeneity of two Security Variances

The test is conducted by following a three-step procedure :

(1) A desired level of confidence is selected.

(2) For each security in the sample the F-statistic shown in equation (8.3) is calculated .

(3) Each F-statistic value is compared with the critical test value and the acceptance or rejection of the hypothesis concerning the homogeneity of the variances is decided.

The Difference between two Security Means, Populations with Common Variance

The testing process is :

(1) A desired level of confidence is selected.

(2) For each security in the sample the t-statistic represented by equation (8.5) is computed.

(3) A simple comparison between each value of the t-statistic and the critical test value is performed in order to decide the validity of the hypothesis regarding the intertemporal stationarity of the security means.

The Difference between two Security Means, Populations with Unequal Variances

Since it is assumed equal number of observations the t-statistic of this test is equal to the t-statistic to the previous test (see Section 8.10). Therefore in this case such a test was not performed.

Homogeneity of two Covariance Matrices of Security Returns

The testing process is as follows :

(1) A desired level of confidence is selected.

(2) For each master group of securities, each subgroup and each subperiod, a sample covariance is computed.

(3) For each group of securities and each subgroup, the chi-square statistic described in equation (8.9), is estimated.

(4) By comparing in each master group and in each subgroup the estimated chi-square statistic with the critical test value a conclusion concerning the intertemporal stationarity of the covariance matrix is derived.

Homogeneity of two Correlation Matrices of Security Returns

The testing process can be described as follows :

(1) A desired level of confidence is chosen.

(2) For each master group of securities, each subgroup and each subperiod, a sample correlation matrix is calculated.

(3) For each group of securities and each subgroup the chi-square statistic, given by equation (8.11) is computed.

(4) By comparing in each master group and in each subgroup the computed chi-square statistic with a critical test value a decision concerning the acceptance or rejection of the intertemporal stationarity of the correlation is produced.

Table 9.5 shows the percentage of securities having stationary variance through the sample period 1/1972 - 12/1981 and the percentage of securities having stationary mean returns through the same sample period.
Table 9.6 (9.7) presents the percentage of securities having stationary variances through the sample period 11/1956 - 10/1981 (11/1956 - 12/1981) and the perecentage of securities having stationary mean returns through the same sample period.

Table 9.8 provides the chi-square test values for the intertemporal stationarity of the covariance matrix, using sample A,while Tables 9.9 and 9.10 give the results of the chi-square test for the intertemporal stationarity of the covariance matrix utilizing sample B.

Finally, Table 9.11 lists the results of the chi-square for the intertemporal stationarity of the correlation matrix using sample A, whereas Tables 9.12 and 9.13 contain the results of the chi-square test for the intertemporal stationarity of the correlation matrix utilizing sample B.

Table 9.5 Intertemporal stationarity of the security variances and the security mean returns using two nonoverlapping subperiods of length 60.
Sample A : Period : 1/1972-12/1981, Number of securities : 672 .

SUBPERIODS	PERCENTAGE OF SECURITIES WITH INTERTEMPORAL STATIONARY VARIANCE[1]	PERCENTAGE OF SECURITIES WITH INTERTEMPORAL STATIONARY MEAN RETURN[2]
1/1972 - 12/1976 and 1/1977 - 12/1981	49	100

1,2 The assumed level of confidence is 99% .

Table 9.6 Intertemporal stationarity of the security variances and the security mean returns using four nonoverlapping subperiods of length 75 . Sample B : Period : 11/1956-12/1981 , Number of securities : 200 .

SUBPERIODS	PERCENTAGE OF SECURITIES WITH INTERTEMPORAL STATIONARY VARIANCE[1]	PERCENTAGE OF SECURITIES WITH INTERTEMPORAL STATIONARY MEAN RETURN[2]
11/1956 - 1/1963 and 2/1963 - 4/1969	87	100
5/1969 - 7/1975 and 8/1975 - 10/1981	77	100

1,2 The assumed level of confidence is 99% .

Table 9.7 Intertemporal stationarity of the security variances and the security mean returns using two nonoverlapping subperiods of length 151 . Sample B : Period 11/1956-12/1981 , Number of securities : 200 .

SUBPERIODS	PERCENTAGE OF SECURITIES WITH INTERTEMPORAL STATIONARY VARIANCE[1]	PERCENTAGE OF SECURITIES WITH INTERTEMPORAL STATIONARY MEAN RETURN[2]
11/1956 - 5/1969 and 6/1969 - 12/1981	49	100

1,2 The assumed level of confidence is 99% .

Table 9.8 A chi-square test for the intertemporal stationarity of the covariance matrix of security returns ,using two nonoverlapping subperiods of length 60 . Sample A : Period: 1/1972-12/1981, Number of securities :672.

		GROUPS CONTAINING 10 SECURITIES	GROUPS CONTAINING 21 SECURITIES	GROUPS CONTAINING 42 SECURITIES
SUBPERIODS	NUMBER OF GROUP	DEGREES OF FREEDOM:55 CRITICAL VALUE:82.2 [1] x^2	DEGREES OF FREEDOM:231 CRITICAL VALUE:283.9 x^2	DEGREES OF FREEDOM:903 CRITICAL VALUE:1004.8 x^2
1/1972- 12/1976 and 1/1977- 12/1981	1	112.0	314.0	1175.1
	2	114.2	356.0	1268.5
	3	125.3	367.3	1116.7
	4	104.8	352.3	1188.9
	5	98.1	421.7	1184.7
	6	102.1	352.0	1230.1
	7	133.8	391.6	1303.0
	8	107.6	361.8	1201.9
	9	130.5	351.9	1211.6
	10	109.1	361.0	1293.0
	11	107.9	321.4	1102.7
	12	187.7	395.9	1297.6
	13	127.8	363.2	1285.1
	14	120.6	441.1	1312.0
	15	132.4	332.1	1231.2
	16	129.2	351.5	1256.9

1 The null hypothesis that the covariance matrix of security returns is intertemporally stationary is rejected in all the cases at the 99% level of confidence .

Table 9.9 A chi-square test for the intertemporal stationarity of the covariance matrix of security returns , using four nonoverlapping subperiods of length 75 . Sample B : Period : 11/1956-12/1981 , Number of securities: 200 .

		GROUPS CONTAINING 10 SECURITIES	GROUPS CONTAINING 20 SECURITIES	GROUPS CONTAINING 40 SECURITIES
SUBPERIODS	NUMBER OF GROUP	DEGREES OF FREEDOM:55 CRITICAL VALUE:82.2[1] / x^2	DEGREES OF FREEDOM:210 CRITICAL VALUE:260.6 / x^2	DEGREES OF FREEDOM:820 CRITICAL VALUE: 917.1 / x^2
11/1956- 1/1963 and 2/1963- 4/1969	1	102.8	285.1	1051.6
	2	122.1	292.9	1028.4
	3	101.1	274.7	949.8
	4	133.6	287.7	991.6
	5	93.6	307.4	1069.3
5/1969- 7/1975 and 8/1975- 10/1981	1	113.9	293.9	989.6
	2	120.3	316.2	1151.1
	3	118.4	324.6	1039.1
	4	111.7	305.5	1023.5
	5	99.3	319.5	1039.3

1 The null hypothesis that the covariance matrix of security returns is intertemporally stationary is rejected in all the cases at the 99% level of confidence .

Table 9.10 A chi-square test for the intertemporal stationarity of the covariance matrix of security returns, using two nonoverlapping subperiods of length 151. Sample B : Period : 11/1956-12/1981, Number of securities: 200 .

		GROUPS CONTAINING 10 SECURITIES	GROUPS CONTAINING 20 SECURITIES	GROUPS CONTAINING 40 SECURITIES
SUBPERIODS	NUMBER OF GROUP	DEGREES OF FREEDOM:55 CRITICAL VALUE:82.2[1] / x^2	DEGREES OF FREEDOM:210 CRITICAL VALUE:260.6 / x^2	DEGREES OF FREEDOM:820 CRITICAL VALUE:917.1 / x^2
11/1956-5/1969 and 6/1969-12/1981	1	274.0	632.2	1720.3
	2	266.9	745.3	1461.6
	3	216.3	483.3	1385.1
	4	428.4	581.4	1435.0
	5	235.9	583.1	1334.5

1 The null hypothesis that the covariance matrix of security returns is intertemporally stationary is rejected in all the cases at the 99% level of confidence.

Table 9.11 A chi-square test for the intertemporal stationarity of the correlation matrix of security returns, using two nonoverlapping subperiods of length 60. Sample A : Period: 1/1972-12/1981, Number of securities:672 .

		GROUPS CONTAINING 10 SECURITIES	GROUPS CONTAINING 21 SECURITIES	GROUPS CONTAINING 42 SECURITIES
SUBPERIODS	NUMBER OF GROUP	DEGREES OF FREEDOM:45 CRITICAL VALUE:69.9[1] / x^2	DEGREES OF FREEDOM:210 CRITICAL VALUE:260.6 / x^2	DEGREES OF FREEDOM:861 CRITICAL VALUE:960.4 / x^2
1/1972-12/1976 and 1/1977-12/1981	1	8.8	76.4	616.6
	2	7.6	88.8	689.2
	3	10.8	88.5	598.9
	4	8.9	83.8	641.3
	5	9.2	94.3	693.2
	6	7.9	84.4	658.7
	7	12.0	89.4	656.6
	8	10.7	83.5	642.1
	9	11.7	82.4	651.7
	10	7.6	85.2	661.8
	11	10.6	79.8	612.0
	12	10.0	88.2	681.2
	13	10.9	86.8	656.3
	14	8.9	93.0	623.7
	15	11.6	81.1	632.0
	16	11.4	78.8	645.0

1 The null hypothesis that the correlation matrix of security returns is intertemporally stationary is accepted in all the cases at the 99% level of confidence.

Table 9.12 A chi-square test for the intertemporal stationarity of the correlation matrix of security returns,using four nonoverlapping subperiods of length 75 . Sample B : Period : 11/1956-12/1981 , Number of securities 200.

		GROUPS CONTAINING 10 SECURITIES	GROUPS CONTAINING 20 SECURITIES	GROUPS CONTAINING 40 SECURITIES
SUBPERIODS	NUMBER OF GROUP	DEGREES OF FREEDOM:45 CRITICAL VALUE:69.9[1] / x^2	DEGREES OF FREEDOM:190 CRITICAL VALUE:238.2 / x^2	DEGREES OF FREEDOM:780 CRITICAL VALUE:874.8 / x^2
11/1956-1/1963 and 2/1963-4/1969	1	6.81	54.1	457.7
	2	8.04	59.8	449.3
	3	10.2	58.8	444.2
	4	7.63	54.0	427.6
	5	7.38	58.4	445.4
5/1969-7/1975 and 8/1975-10/1981	1	7.3	55.0	428.6
	2	7.2	57.8	456.1
	3	9.2	59.1	453.1
	4	7.7	56.5	442.3
	5	6.6	62.5	452.4

1 The null hypothesis that the correlation matrix of security returns is intertemporally stationary is accepted in all the cases at the 99% level of confidence .

Table 9.13 A chi-square test for the intertemporal stationarity of the correlation matrix of security returns, using two nonoverlapping subperiods of length 151. Sample B : Period : 11/1956-12/1981, Number of securities :200 .

		GROUPS CONTAINING 10 SECURITIES	GROUPS CONTAINING 20 SECURITIES	GROUPS CONTAINING 40 SECURITIES
SUBPERIODS	NUMBER OF GROUP	DEGREES OF FREEDOM:45 CRITICAL VALUE:69.9[1] / x^2	DEGREES OF FREEDOM:190 CRITICAL VALUE:238.2 / x^2	DEGREES OF FREEDOM:780 CRITICAL VALUE:874.8 / x^2
11/1956-5/1969 and 6/1969-12/1981	1	4.6	36.3	246.4
	2	6.1	39.6	254.2
	3	5.9	30.9	233.0
	4	7.1	35.0	240.3
	5	4.9	38.0	230.5

1 The null hypothesis that the correlation matrix of security returns is intertemporally stationary is accepted in all the cases at the 99% level of confidence .

9.3 Description of the Empirical Results

The results of Table 9.5 indicate that about half of the securities have stationary variance during the sample period 1/1972 - 12/1981, while all the securities have stationary mean returns during the same period. Therefore one may conclude the intertemporal stationarity of sample A's security mean returns.

Furthermore from Table 9.6 (9.7) one observes that more (less) than half of the securities have stationary variance during the sample period 11/1956 - 10/1981 (11/1956 - 12/1981), whereas all the securities have stationary mean returns during the same period. Consequently the security mean returns of sample B are intertemporally stationary.

The difference percentage concerning the stationarity of security variance may be due to the abnormal fluctuations of the security returns during the 1974 - 75 period.

It was mentioned in Section 8.10 that the assumption of equal population security variances is relatively unimportant when the test of the difference between two security mean returns assumes equal number of observations. However, t-tests for the intertemporal stationarity of the security mean returns were performed by assuming unequal number of observations. The results showed again that the security mean returns are intertemporally stationary.

Table 9.8 shows that the chi-square statistics calculated by using groups of 10, 21 and 42 securities are greater than the critical values for the chi-square distribution at the 99% level of confidence. These findings indicate that the covariance matrix of security returns is not stationary during the sample period 1/1972 - 12/1981. It is also clear that such conclusion is independent of the number of securities included in the groups.

The results reported in Tables 9.9 and 9.10, derived by using groups of 10, 20 and 40 securities, indicate that the calculated chi-square values are greater than the critical values for the chi-square distribution at the 99% level of confidence. Therefore it can be deduced that the covariance matrix of security returns is not stationary during the sample period 11/1956 - 12/1981. Also it is inferred that such a conclusion is independent from the number of securities included in the group and from the number of observations per security.

Although the samples A and B use different securities per group and different sample sizes per security the results of Table 9.7 are in line with the results of Tables 9.9 and 9.10.

From Table 9.11 one notes that the chi-square statistics calculated by utilizing groups of 10, 21 and 42 securities are not significant at the 99% level of confidence. As a consequence the correlation matrix of security returns is stationary during the sample period 1/1772 - 12/1981, and this conclusion is independent of the security group size.

The empirical results of Tables 9.12 and 9.13 show that the chi-square statistics computed by using groups of 10, 20 and 40 securities are not significant at the 99% level of confidence. Thus the correlation matrix of security returns is stationary during the sample period 1/1956 - 12/1981 and this conclusion is independent of the security group size and from the number of observations per security.

The results of Table 9.11 taken in conjunction with those of Tables 9.12 and 9.13 seems to indicate quite convincingly that the correlation matrix is intertemporally stationary despite the fact that both samples use different securities and different sample sizes per security.

Lastly, the results of the Tables 9.8, 9.9 and 9.10 contrast sharply with the results on the intertemporal stationarity of the correlation matrix shown in Tables 9.11, 9.12 and 9.13.[1]

1. The intertemporal stationarity of security mean returns, the covariance matrix of security returns and the correlation matrix of security returns was also examined for the sample B and the subperiods 11/1956 - 2/1965 and 3/1965 - 6/1973, 3/1965 - 6/1973 and 7/1973 to 10/1981. The results also indicated the intertemporal stationarity of the mean returns, the intertemporal stationarity of the correlation matrices, but the intertemporal non-stationarity of the covariance matrices.

The conclusions of the assumption concerning the intertemporal stationary distributions of security returns was derived utilizing a large number of groups of different sizes and subperiods of 60, 100 and 151 monthly observations. Thus by taking into consideration both samples the reported results are more powerful and reliable.

Finally, the conclusions derived by verifying empirically the intertemporal stationarity of the security return distributions can be summarized as follows :

(1) The security mean returns are intertemporally stationary.

(2) The covariance matrices of security returns are not intertemporally stationary.

(3) The correlation matrices of security returns are intertemporally stationary.

9.4 The Implications of the Empirical Results

Although the assumption about the intertemporal stationarity of security return distrubtions was assumed by all the studies involved empirical examinations of asset pricing models, only one study provided direct empirical results of such an assumption.
Gibbons (1981) investigated empirically the intertemporal stationarity of the covariance and correlation matrices utilizing one group of industry portfolios and one group

of bond portfolios. Even though his results apply for the New York Stock Exchange he also rejected the intertemporal stationarity of the covariance matrix and he accepted the intertemporal stationarity of the correlation matrix. Gibbons, however, did not examine the intertemporal stationarity of the portfolio mean returns.

When the covariance matrix of security returns is not intertemporally stationary the assumption of the stationarity in the joint distribtution of security returns is violated. In this case by factor analyzing the covariance matrix , heteroscedastic specific variances are produced. This implies the asympotic inefficiency of the factor loadings and thus questions are raised concerning the results based on the estimated factor loadings.

Fortunately, the heteroscedastic security specific variances can be corrected by utilizing the correlation matrix of security returns. This can be achieved since the entries of the correlation matrix are not dependent on the security specific variances (see equation(7.5)) . Furthermore factor analysis can be performed on the correlation matrix. By factor analyzing an intertemporal stationary correlation matrix the estimated factor loadings will fulfil all the desired properties (see Section 7.4). Consequently tests for statistical influence based on these estimates will be valid .

However, the violation of the intertemporal stationary distribution of security returns assumption has more general implications for the portfolio theory. These are :

(A) Markowitz Model

Tests concerning the original Markowitz model are based upon estimates of a covariance matrix of security returns. Such tests rely on the assumption that ex-ante distributions can be well approximated by ex-post distributions. However, if the covariance matrix of security returns is not intertemporally stationary this assumption is violated.

In the absence of the intertemporal stationarity in the distribution of security returns the ex-post efficient frontier in mean-standard deviation space will be either shifted to the right or to the left of the ex-ante efficient frontier. In this case the following consequences emerge :

(1) The risk of the efficient portfolios will be either over estimated or under estimated and hence the investors will have a misleading picture regarding the expected risk of their portfolios.

(2) Inefficient portfolios may be identified as effecient and vice-versa.

(B) Capital Asses Pricing Model

Fama (1976) pointed out that the intertemporal stationary distribution assumption is compatible and gives strength to the assumption of homogeneous expectations on which is based the C.A.P.M. (see Fama (1976) p.344).

In view of the homogeneous expectation assumption all investors in the market agree about the expected return and the variability of the returns on all securities at the end of some period of time which is also identical for all investors. That is all investors face the same ex-ante efficient frontier. This in turn implies that the problem of choosing a portfolio of risky securities is independent of their attitude towards risk (Separation Theorem). Hence the absence of intertemporal stationarity in the distribution of security returns shows that the homogeneous expectation assumption and its implications are not sensible approximations to the real world.

Specifically, Rosenberg and Ohlson (1976) stated :
"It is unfortunate that the assumptions of separability and of stationarity and serially independent returns do not appear to be sufficiently rich to admit realistic conclusion on asset price behaviour" (p.401).

Roll (1977) proved that the (N x 1) investment proportions vector defining an arbitrary minimum standard deviation portfolio, call it M_1, can be expressed as :

$$X_{M_1} = V^{-1} (R \; i) A^{-1} \begin{pmatrix} r_{M_1} \\ 1 \end{pmatrix}$$

where

V = the (N x N) covariance matrix of security returns.

R = the (N x 1) column vector of security expected returns.

i = the (N x 1) unit vector.

A = a (2 x 2) information matrix whose entries are dependent on V,R and i.

r_{M_1} = the expected return of the portfolio M_1.

If V changes through time, then X_{M_1} changes through time.

Thus it is possible to identify a portfolio (proxy), as ex-ante efficient while it is not. In this case it should be falsely accepted the validity of an ex-ante exact linear relationship. On the other hand, one may identify a portfolio as ex-ante inefficient while it is not. In such a case it should be incorrectly concluded the rejection of an ex-ante exact linear relationship.

The violation of the stationary distribution assumption may also produce a minimum standard deviation portfolio which is ex-post uncorrelated with the proxy, while in reality there exists a different portfolio which is ex-ante uncorrelated with the proxy. Therefore the violation of the intertemporal stationary distribution assumption produces problems in testing the efficiency of a market proxy. If the market portfolio was observable then the same problems would emerge if the distributions of security returns were not stationary through time.

Roll (1976) expressed his view concerning this point as follows :

"The concern throughout will be with the ex-ante efficiency of a particular pre-selected vector of investment proportions. This is primarily a small-sample problem because the sample efficient set approaches the population efficient set asympotically. Of course, there must be an implicit assumption of stationarity of the population. If the ex-ante means and covariances are changing as fast as the sample size,no test of market proxy efficiency would be uuambiguous." (p.45).

Although the importance of the intertemporal stationarity of security returns was recognised by many studies, it is surprising to see only one direct empirical test in the published literature.

(C) Market Model and Ex-Post Form of the Capital Asset Pricing Model

The assumption of the intertemporal stationarity of security returns has also played an important role in the estimation of the security beta coefficients and the security specific variances. In fact this assumption constitutes the necessary condition to estimate the security beta coefficients and the security specific variances by the ordinary least squares method (O.L.S.M.).

All the previous tests of the C.A.P.M. and some tests concerning the efficiency of the capital market utilized the market model or the ex-post form of the C.A.P.M. and the O.L.S.M. to estimate the beta coefficients and the security specific variances. The market model and the ex-post form of the C.A.P.M. are static models. Therefore, when such models are employed to test these hypotheses using time series data, it is necessary to rely on the intertemporal stationarity distribution assumption. Unfortunately, the previous tests were based on the intertemporal stationarity distribution assumption, but no attempt was made to empirically examine such an assumption. If this assumption is violated the following consequences occur :

(1) The procedure employed to derive the ex-post of the C.A.P.M. is incorrect and thus by following such a procedure one cannot transform the C.A.P.M. (S.R.R.E.L.R.) into a testible relationship.

(2) The ordinary least squares estimates tend to be less accurate and hence standard statistical tests of significance are invalid.

Therefore empirical tests assuming stationary probability distributions of security returns in a non-stationary world should be intrepeted with caution. In my knowledge the majority the U.K. tests, concerning the empirical investigation of the C.A.P.M., the estimation of the beta coefficients, the security specific variances and the efficiency of the capital market implicity assumed the stationarity distrubtion assumption. However, none of these tests have taken into consideration the consequences of non-stationary probability distributions.

If the parameters of the distributions of security returns are not stationary through time then attention has to be given to models with non-stationary parameters. Such models can be found in studies by Barry and Winkler (1976), Barry (1978), Fabozzi and Francis (1978), Brenner and Smibt (1977), Chen (1981c), Lee and Chen (1980), Chen and Keown (1981a, 1981b), Alexander and Benson (1982) and Sunber (1980).

9.5 Conclusions

The rejection of the intertemporal stationarity of the covariance matrix of security returns and the acceptance of the intertemporal stationarity of the correlation matrix leads to the conclusion that the correlation matrix has to be used in the A.P.M.'s tests. In addition the results derived by invoking the intertemporal stationary distribution assumption, when such an assumption is violated, must be interpreted with caution.

Finally, the O.L.S.M. is not an appropriate method to be used to estimate the beta coefficients and the specific variances if the joint distribution of security returns is not intertemporally stationary.

CHAPTER 10

THE RELATIONSHIP BETWEEN THE NUMBER OF COMMON FACTORS AND THE GROUP SIZE

This chapter empirically investigates whether the number of factors affecting the security returns is the same across various groups of different sizes and across various groups of the same size.

The chapter begins by presenting the statistical procedure employed and it reports the empirical results This is followed by describing the empirical results and discussing some possible explanations of the results. Finally, the implications of these empirical findings are presented .

To test the relationship between the number of common factors and the group size tests 7 and 8 described in Table 8.4 are utilized. That is, a test for the complete independence of the correlation matrix of security returns and Rao's test for the goodness of fit of the factor model. The former test is a precondition of the latter; If the correlation matrices contain significant non-diagonal entries then one continues to perform the latter test.

The two previously mentioned tests are separately adopted for each of the two samples (A and B) considered in this study.

Sample A of 672 securities, used to generate sixteen random master groups of size 42 and each master group is broken down into seven subgroups containing 5, 10, 15, 21, 26, 31 and 36 securities, respectively.

Furthermore the 200 securities of sample B are randomly classified into five master groups of size 40 and from each master group are drawn seven subgroups containing 5, 10, 15, 20, 25, 30 and 35 securities respectively.

10.1 Statistical Procedure and the Empirical Results

To examine the adequancy of the correlation matrix for factor analysis and the relationship between the number of factors and the group size the necessary testing procedures were followed. Below is provided a description for each procedure.

Completë Independence of the Non-Diagonal Entries of the Correlation Matrix

This test requires a four step procedure. That is :

(1) A desired level of confidence is chosen.

(2) For each group of securities the correlation matrix is

estimated and its determinant is calculated.

(3) For each group of securities the chi-square test statistic is calculated by virtue of equation (8.13).

(4) A comparison between the chi-square value and the critical test value is made and a conclusion concerning the acceptance or rejection of the hypothesis is presented.

Rao's Test for the Goodness of Fit of the Factor Model

When Rao's test for the goodness of fit of the factor model is employed there are two different methods of extracting the number of factors.

The first method suggested by Rao and it can be briefly explained as follows :

Initially an arbitrary number of factors, say K, is specified and the solution for Rao's factor analysis, conditional on the correlation matrix generated by exactly K factors is derived . If the chi-square value (see equation (8.24)) is not significant then more than K factors are required to explain the variability of security returns. In this case the process is terminated when a significant chi-square value is found.

The second method for estimating the number of common factors considers different values of K sequentially starting with K = 1.

There are cases, however, where Rao's procedure gives misleading conclusions. To illustrate a case one must consider a subgroup of size 10 and apply both the sequential and Rao's procedure. Indeed Table 10.1 shows that the sequentially procedure implies the existence of three factors having influence on security returns, while Rao's procedure implies the existence of five. Since the aim of factor analysis is to explain the intercorrelations between a large number of variables (security rate of returns) by introducing a minimal number of factors it can be deduced that the sequential pricedure is most appropriate. Therefore for testing the relationship between the number of factors and the group size the sequential procedure is adopted.

To test the goodness of fit of the factor model using the sequential procedure, the following steps are followed :

(1) For a group of individual securities the number of factors is set equal to 1. Then Rao's factor analytic method is performed on the symmetric matrix GRG generated by exactly one common factor and a chi-square value is obtained (G = a diagonal matrix whose diagonal entries are given by equation (8.22) and R is the sample correlation matrix).

(2) If the value of the chi-square is not significant, a single-factor model is required to explain the variability of the security returns into this group. If it is significant, the number of factors is set equal to 2. Then Rao's factor analytic method is performed

Table 10.1. Rao's testing procedure versus the sequential procedure.

	NUMBER OF FACTORS	x^2	DEGREES OF FREEDOM	CRITICAL VALUE (S) FOR THE x^2 DISTRIBUTION AT THE 99% LEVEL OF CONFIDENCE
Rao's [1] Testing Procedure	5	13.1	5	15.0 * [2]
Sequential Procedure	1	158.3	35	57.3
	2	61.3	26	45.6
	3	24.6	18	34.8 *

1 The size of the group used is 10.

2 The null hypothesis is that K factors are required to explain the variability of security returns. Asterisks indicate that the null hypothesis is accepted at the 99% level of confidence.

on the symmetric matrix GRG generated by exactly two factors and two chi-square values are computed.

(3) If the second value of the chi-square is not significant two factors are required to expalin the security returns in the group.
If it is significant the number of factors is increased one by one until a model is found that gives a satisfactory fit with the minimum number of factors.

If the only aim of the research is a test of the significance of the number of factors then the sequentially procedure requires a small number of iterations. To illustrate the case one can utilize a group containing 21 securities and apply Rao's factor analyitic technique by considering different numbers of iterations. Indeed Table 10.2 shows that the conclusion regarding the appropriate number of factors is independent of the number of iterations.

From Table 10.2 it is clear that if the hypothesis concerning the appropriate number of factors is accepted when the number of iterations is small, then it will be definitely accepted as the number of iterations increases. For the factor analytic tests of this study the number of iterations is set equal to 25.

Next Table D.1 in Appendix D indicates that the correlations between the security returns of sample A are significantly different from zero .

Table 10.2 Chi-square test and the number of iterations.

DEGREES OF FREEDOM : 99	CRITICAL VALUE FOR THE x^2 DISTRIBUTION AT THE 99% LEVEL OF CONFIDENCE : 134.6	
NUMBER OF ITERATIONS [1]	x^2	NUMBER OF FACTORS
25	122.7	6
50	120.1	6
100	118.9	6
150	118.5	6
200	118.4	6
300	118.2	6
500	118.1	6

1 The size of group used is 21 .

Moreover, Table D.2 shows that the off-diagonal elements of the correlation matrices of security returns of sample B are significantly different from zero.
Consequently the correlation matrices estimated by using samples A and B are appropriate for factor analysis.

Using sample A it can be seen in Table 10.3 the number of factors emerging via Rao's factor analysis as the group size of securities increases.

Utilizing sample B, Table 10.4 presents the number of factors emerging via Rao's factor analysis as the group size increases.[1]

Finally, the plottings of the number of factors against the group size are presented in Figures 10.1 and 10.2 .

10.2 Description of the Empirical Results

By utilizing sample A and considering each master group with its subgroups, Table 10.3 shows that the appropriate number of factors changes as the group size changes in 92 cases out of 100 (i.e. 410 cases out of 448). In these cases the number of factors increases with the group size. Even if one averages the number of factors it is evident that in almost all the cases the number of factors does not remain the same as the group size increases.

1. A more detailed representation of the results of Tables 10.3 and 10.4 is given in Appendix D.

Table 10.3 The number of factors versus group size.
Sample A : Period: 1/1972-12/1981 , Number of securities : 672 .

	MASTER GROUP 1		MASTER GROUP 2		MASTER GROUP 3		MASTER GROUP 4	
GROUP SIZE	NUMBER OF FACTORS	COMMULATIVE PERCENTAGE OF TOTAL VARIANCE ACCOUNTED FOR BY THE COMMON FACTORS	NUMBER OF FACTORS	COMMULATIVE PERCENTAGE OF TOTAL VARIANCE ACCOUNTED FOR BY THE COMMON FACTORS	NUMBER OF FACTORS	COMMULATIVE PERCENTAGE OF TOTAL VARIANCE ACCOUNTED FOR BY THE COMMON FACTORS	NUMBER OF FACTORS	COMMULATIVE PERCENTAGE OF TOTAL VARIANCE ACCOUNTED FOR BY THE COMMON FACTORS
5	1	67.5	1	60.3	1	55.5	1	49.9
10	3	75.2	2	64.4	1	49.4	1	44.7
15	6	80.7	2	58.7	1	48.8	2	54.2
21	6	74.8	4	66.4	3	58.3	5	65.3
26	7	75.6	6	55.7	6	66.2	8	71.4
31	7	72.4	9	70.3	8	71.3	12	79.8
36	10	75.2	12	72.3	11	75.7	15	81.4
42	13	73.5	15	80.2	13	74.7	16	85.9

Table 10.3
(Continued)

	MASTER GROUP 5		MASTER GROUP 6		MASTER GROUP 7		MASTER GROUP 8	
GROUP SIZE	NUMBER OF FACTORS	COMMULATIVE PERCENTAGE OF TOTAL VARIANCE ACCOUNTED FOR BY THE COMMON FACTORS	NUMBER OF FACTORS	COMMULATIVE PERCENTAGE OF TOTAL VARIANCE ACCOUNTED FOR BY THE COMMON FACTORS	NUMBER OF FACTORS	COMMULATIVE PERCENTAGE OF TOTAL VARIANCE ACCOUNTED FOR BY THE COMMON FACTORS	NUMBER OF FACTORS	COMMULATIVE PERCENTAGE OF TOTAL VARIANCE ACCOUNTED FOR BY THE COMMON FACTORS
5	1	62.3	1	68.8	2	80.3	1	49.2
10	2	61.1	1	52.9	2	71.0	1	46.6
15	3	67.4	4	67.4	3	72.4	1	44.2
21	3	55.8	4	64.5	3	62.4	3	57.0
26	6	67.5	6	67.5	6	74.2	5	61.6
31	6	65.3	10	77.3	9	80.1	6	60.8
36	8	68.5	12	78.1	11	80.4	8	67.9
42	11	73.3	14	80.2	13	81.3	11	70.1

Table 10.3
(Continued)

GROUP SIZE	MASTER GROUP 9 NUMBER OF FACTORS	MASTER GROUP 9 COMMULATIVE PERCENTAGE OF TOTAL VARIANCE ACCOUNTED FOR BY THE COMMON FACTORS	MASTER GROUP 10 NUMBER OF FACTORS	MASTER GROUP 10 COMMULATIVE PERCENTAGE OF TOTAL VARIANCE ACCOUNTED FOR BY THE COMMON FACTORS	MASTER GROUP 11 NUMBER OF FACTORS	MASTER GROUP 11 COMMULATIVE PERCENTAGE OF TOTAL VARIANCE ACCOUNTED FOR BY THE COMMON FACTORS	MASTER GROUP 12 NUMBER OF FACTORS	MASTER GROUP 12 COMMULATIVE PERCENTAGE OF TOTAL VARIANCE ACCOUNTED FOR BY THE COMMON FACTORS
5	1	47.8	1	46.8	1	58.3	1	59.7
10	1	41.0	1	40.0	1	48.3	2	62.0
15	1	40.7	1	43.2	2	52.6	2	58.9
21	3	54.4	3	55.2	3	56.8	3	62.0
26	4	57.3	5	60.7	5	66.1	7	73.7
31	8	70.9	7	65.1	7	69.3	7	70.8
36	11	80.7	9	76.4	8	69.0	7	71.4
42	15	81.9	12	78.3	12	83.3	11	75.6

Table 10.3
(Continued)

	MASTER GROUP 13		MASTER GROUP 14		MASTER GROUP 15		MASTER GROUP 16			
GROUP SIZE	NUMBER OF FACTORS	COMMULATIVE PERCENTAGE OF TOTAL VARIANCE ACCOUNTED FOR BY THE COMMON FACTORS	NUMBER OF FACTORS	COMMULATIVE PERCENTAGE OF TOTAL VARIANCE ACCOUNTED FOR BY THE COMMON FACTORS	NUMBER OF FACTORS	COMMULATIVE PERCENTAGE OF TOTAL VARIANCE ACCOUNTED FOR BY THE COMMON FACTORS	NUMBER OF FACTORS	COMMULATIVE PERCENTAGE OF TOTAL VARIANCE ACCOUNTED FOR BY THE COMMON FACTORS	AVERAGE NUMBER OF FACTORS	COMMULATIVE PERCENTAGE OF TOTAL VARIANCE ACCOUNTED FOR BY THE COMMON FACTORS
5	1	48.1	1	72.6	1	53.9	1	50.4	1	58.6
10	2	60.6	2	71.6	1	56.5	1	41.5	2	55.4
15	3	63.4	3	66.9	2	59.8	1	40.7	2	57.5
21	4	63.6	3	63.5	4	62.4	4	54.9	4	60.1
26	7	72.7	3	61.3	6	71.3	4	52.7	6	65.9
31	7	69.9	4	62.4	9	77.6	4	51.2	8	69.6
36	7	68.4	6	64.4	12	81.3	6	56.8	10	73.0
42	11	78.9	9	74.4	15	85.4	8	58.3	13	77.2

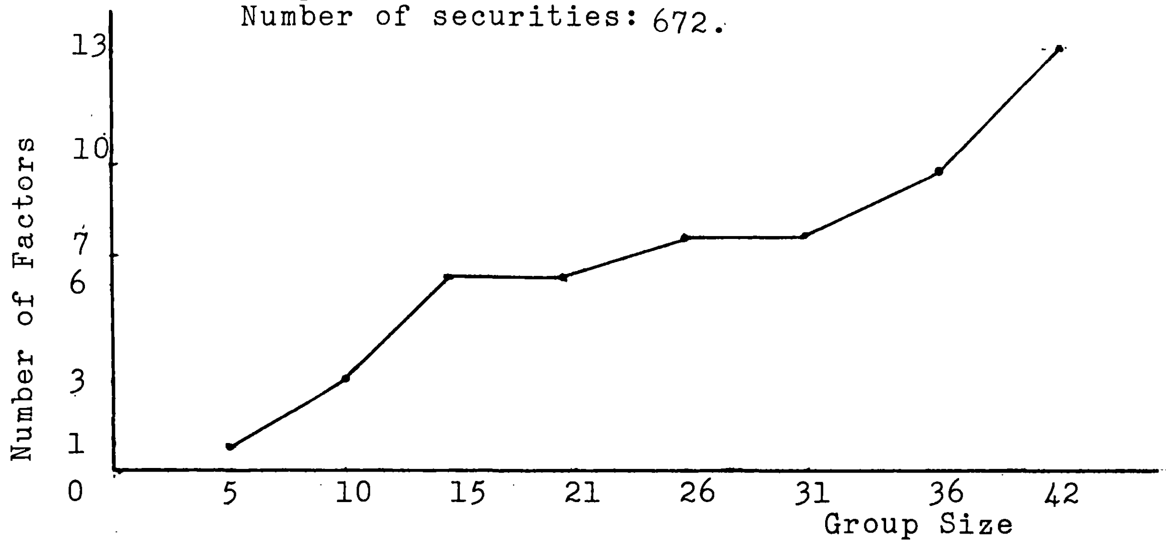
Table 10.4 The number of factors versus the group size.
Sample B : Period : 11/1956-12/1981,Number of securities:200 .

	MASTER GROUP 1		MASTER GROUP 2		MASTER GROUP 3	
GROUP SIZE	NUMBER OF FACTORS	COMMULATIVE PERCENTAGE OF TOTAL VARIANCE ACCOUNTED FOR BY THE COMMON FACTORS	NUMBER OF FACTORS	COMMULATIVE PERCENTAGE OF TOTAL VARIANCE ACCOUNTED FOR BY THE COMMON FACTORS	NUMBER OF FACTORS	COMMULATIVE PERCENTAGE OF TOTAL VARIANCE ACCOUNTED FOR BY THE COMMON FACTORS
5	1	54.6	1	51.4	1	47.7
10	2	62.2	1	48.2	2	62.1
15	4	62.4	2	51.9	4	65.0
20	4	61.6	3	53.8	4	58.6
25	6	67.1	5	60.5	4	54.2
30	7	68.2	7	63.1	6	60.4
35	8	67.4	8	63.7	7	62.3
40	10	65.8	9	63.0	10	65.1

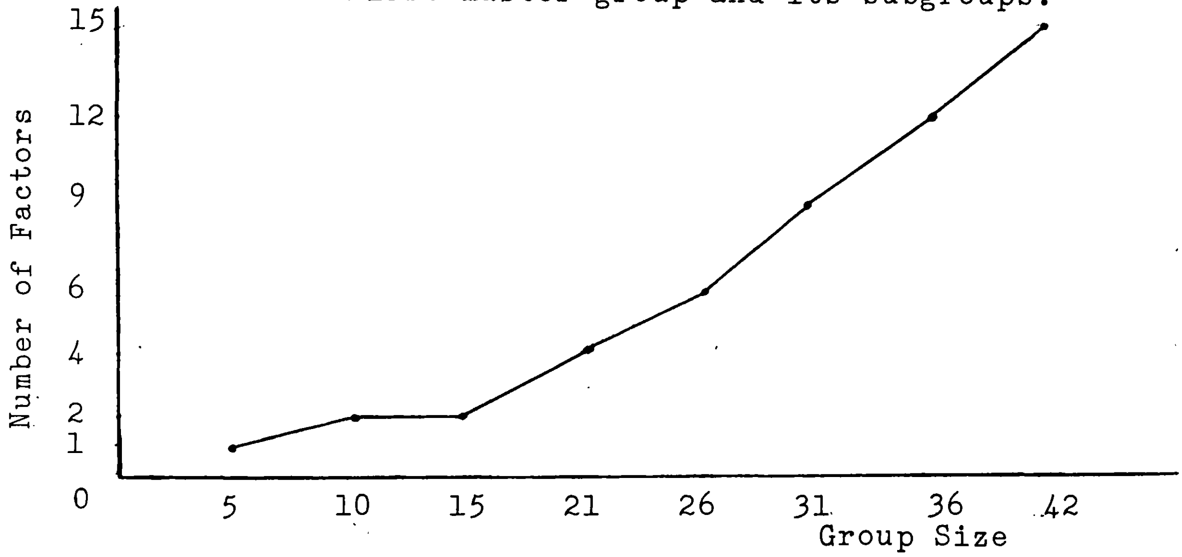
Table 10.4
(Continued)

	MASTER GROUP 4		MASTER GROUP 5			
GROUP SIZE	NUMBER OF FACTORS	COMMULATIVE PERCENTAGE OF TOTAL VARIANCE ACCOUNTED FOR BY THE COMMON FACTORS	NUMBER OF FACTORS	COMMULATIVE PERCENTAGE OF TOTAL VARIANCE ACCOUNTED FOR BY THE COMMON FACTORS	AVERAGE NUMBER OF FACTORS	AVERAGE OF COMMULATIVE PERCENTAGE OF TOTAL VARIANCE ACCOUNTED FOR BY THE COMMON FACTORS
5	1	45.2	2	63.9	1	52.5
10	2	57.7	2	52.7	2	56.5
15	3	55.9	3	59.0	3	60.1
20	5	61.6	3	54.0	4	57.9
25	6	61.7	4	56.9	5	59.1
30	6	61.2	4	55.3	6	61.6
35	8	61.2	5	57.6	7	62.4
40	8	59.9	7	60.5	9	62.8

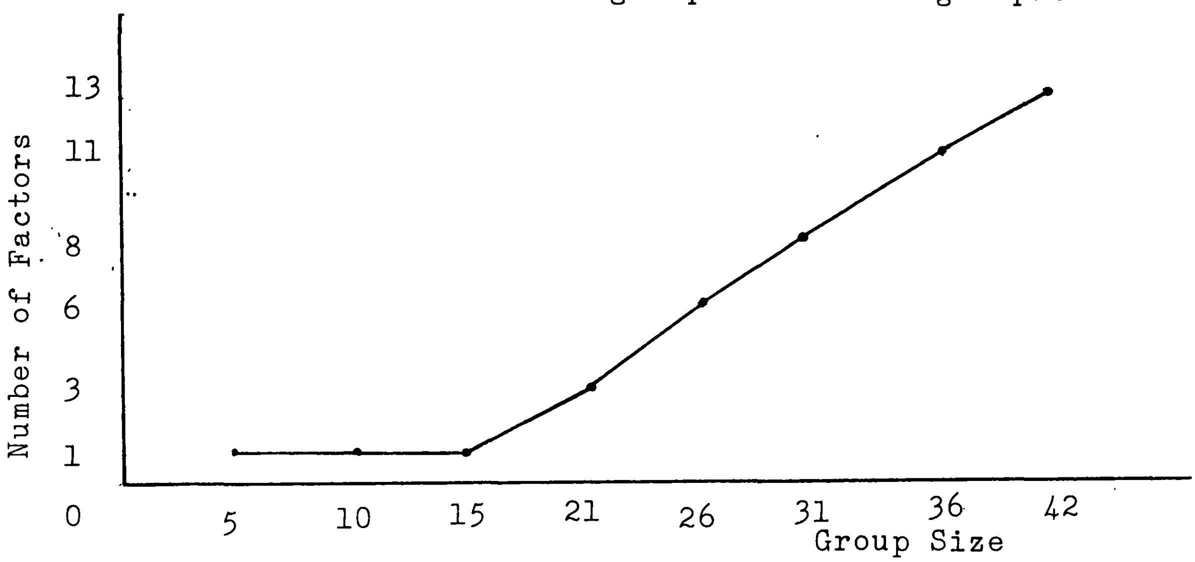
Figure 10.1 Graphical representation between the number of factors and the group size . Sample A:Period:1/1972-12/1981, Number of securities: 672.



First master group and its subgroups.

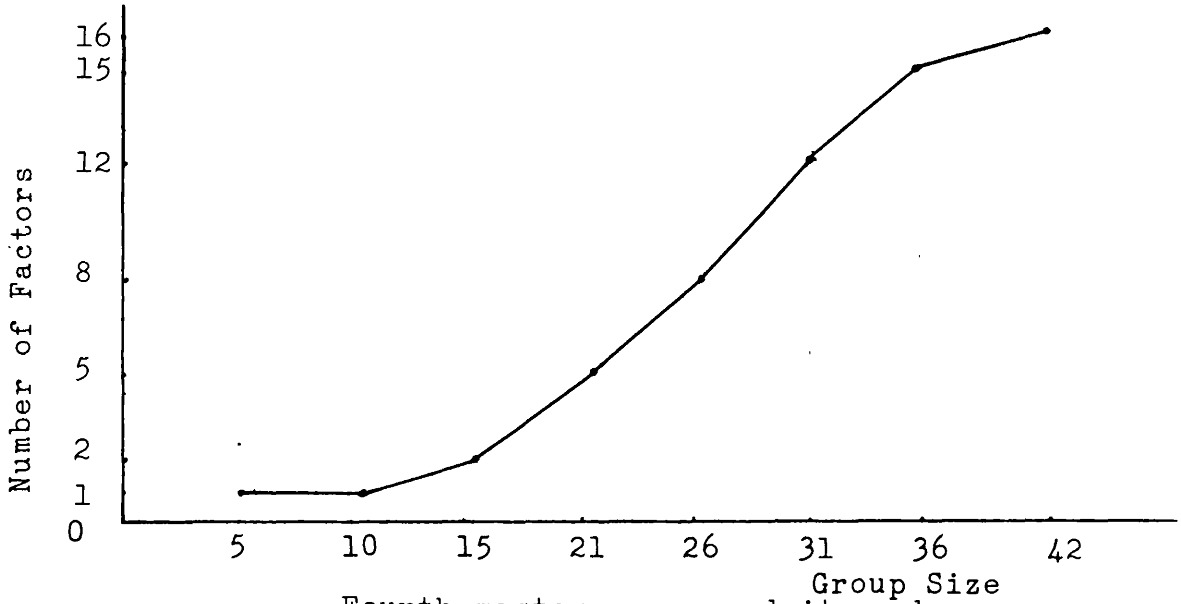


Second master group and its subgroups.

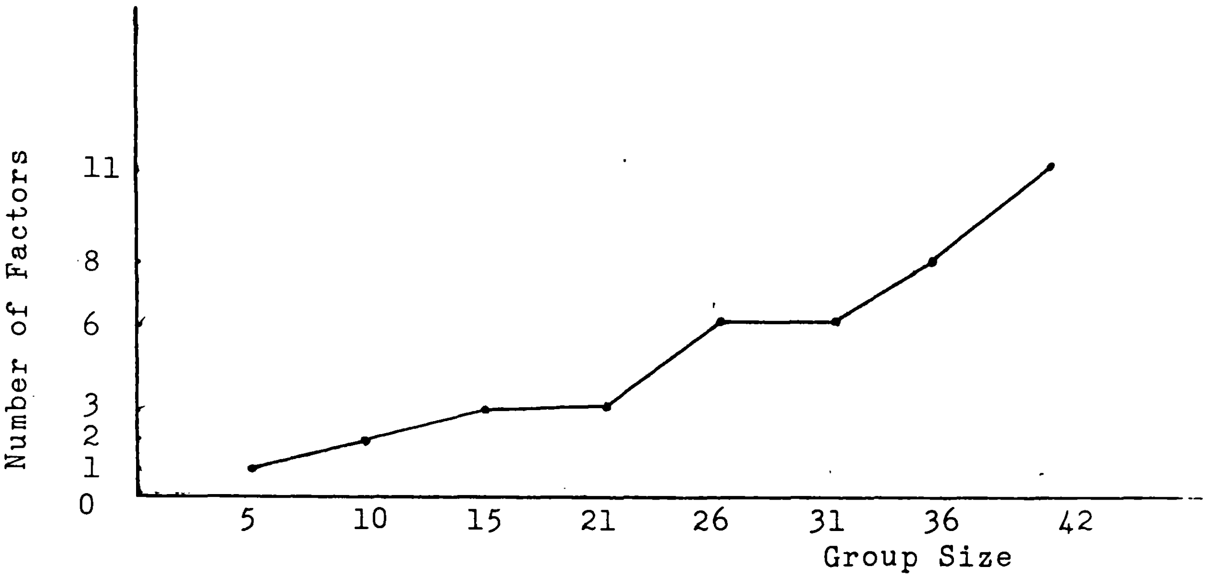


Third master group and its subgroups.

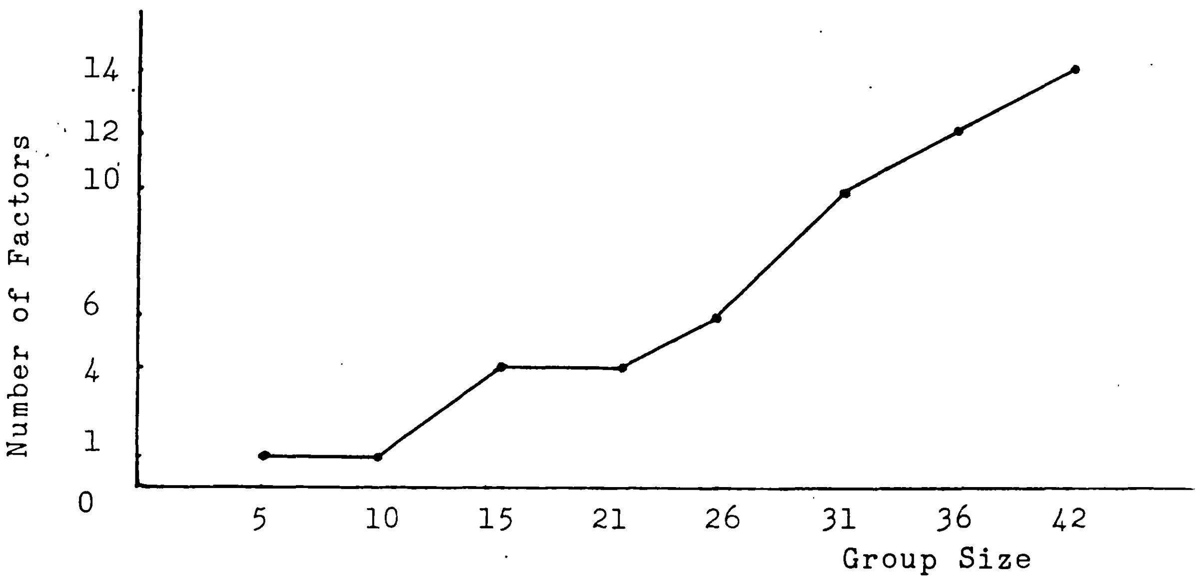
Figure 10.1
(Continued)



Fourth master group and its subgroups.

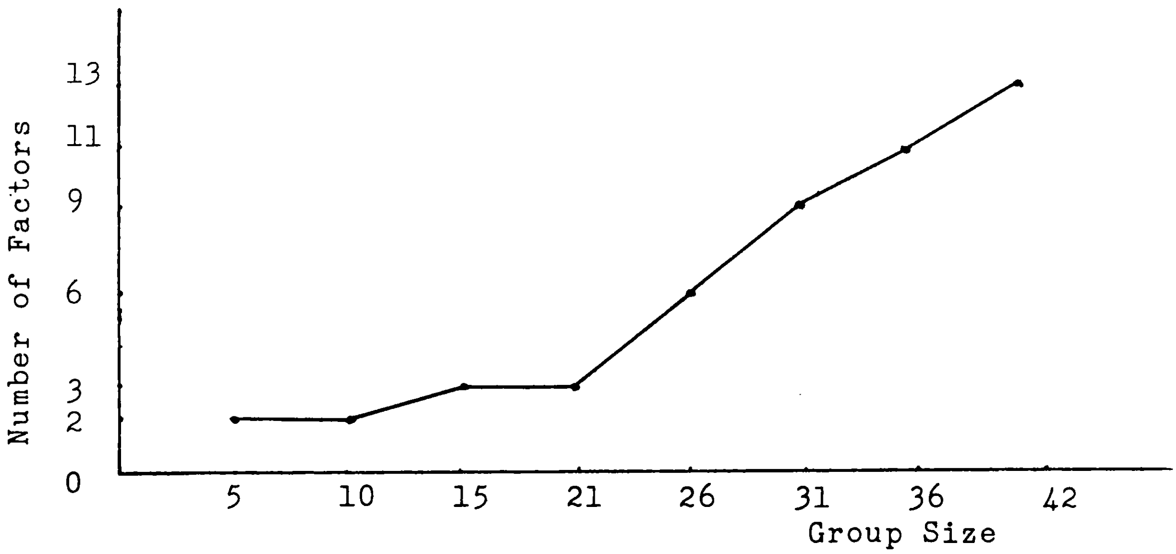


Fifth master group and its subgroups.

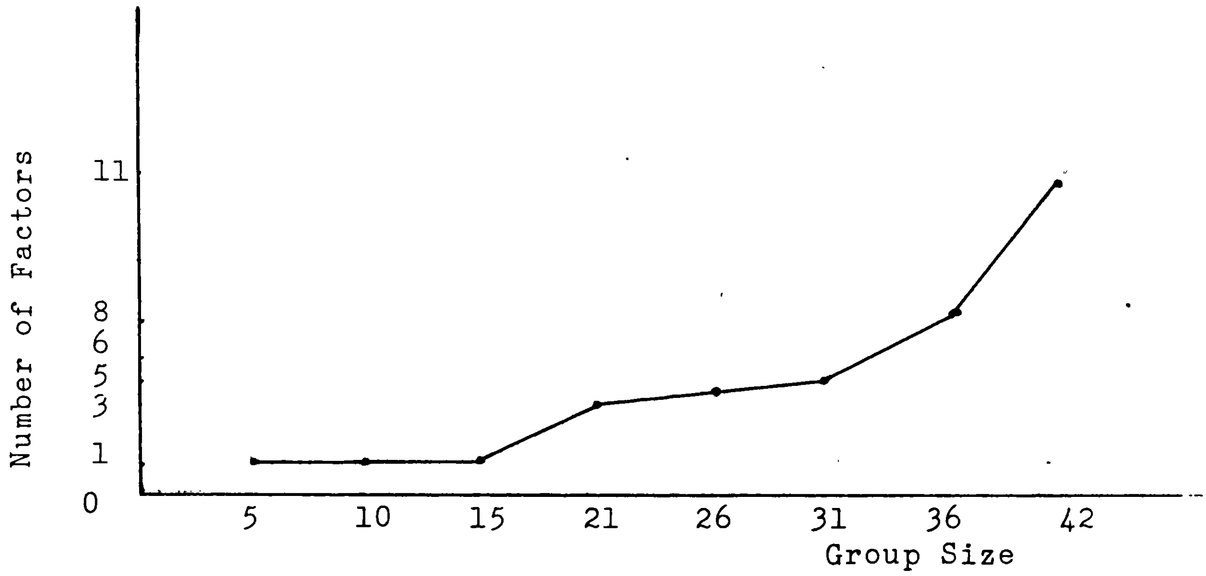


Sixth master group and its subgroups.

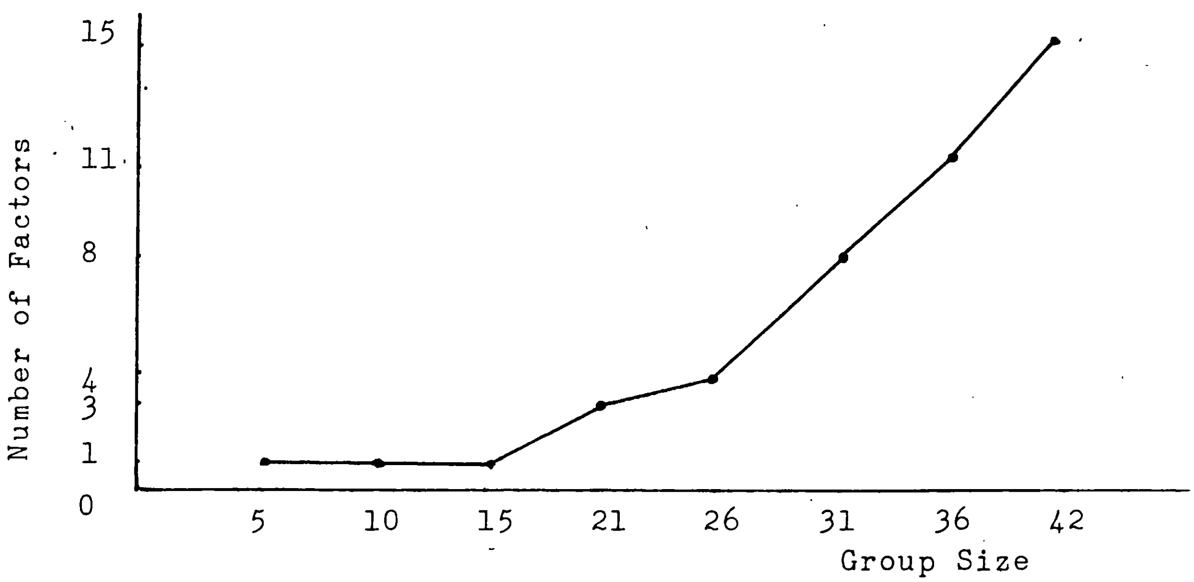
Figure 10.1
(Continued)



Seventh master group and its subgroups.

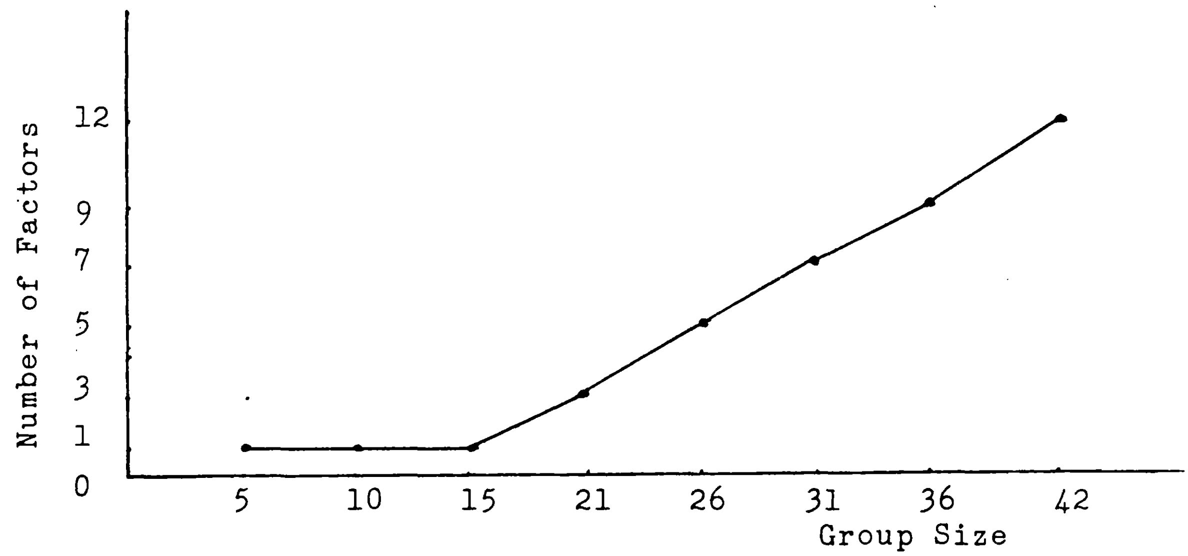


Eighth master group and its subgroups.

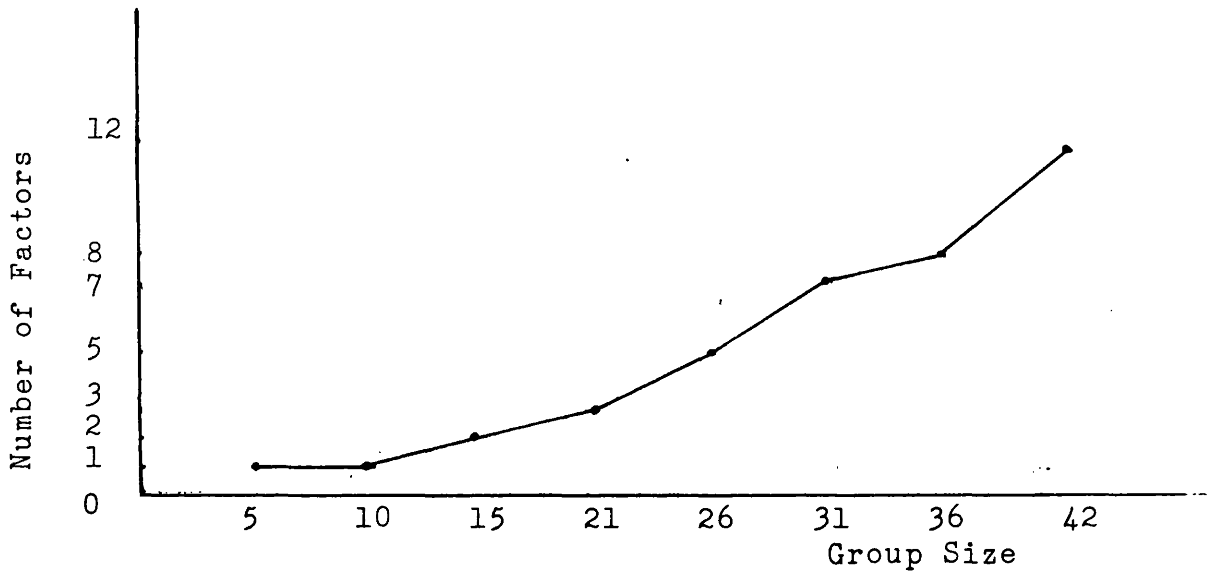


Ninth master group and its subgroups.

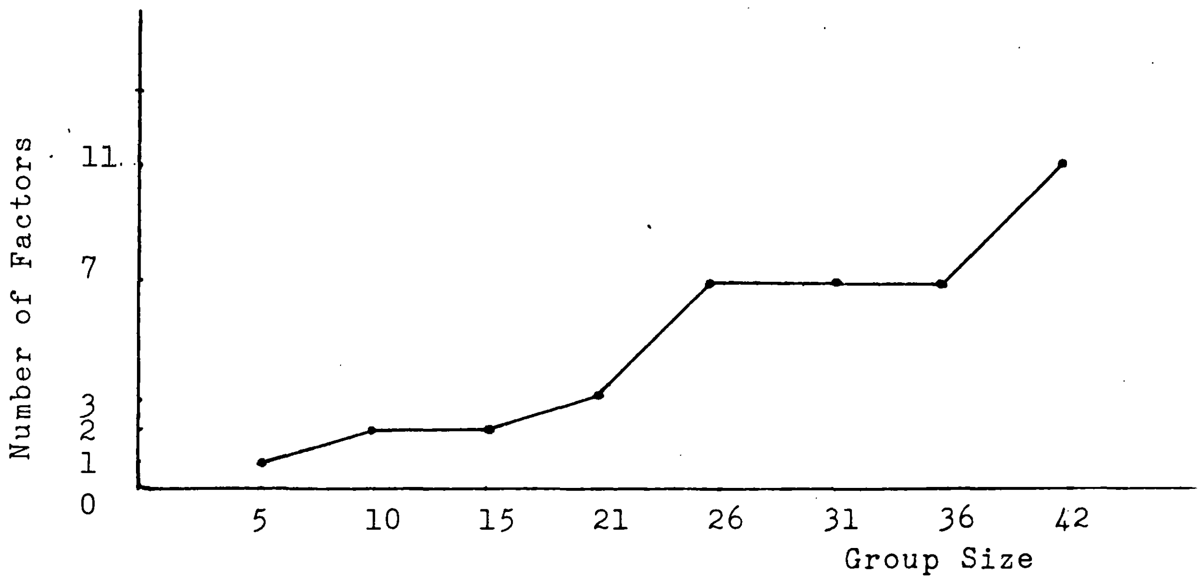
Figure 10.1
(Continued)



Tenth master group and its subgroups.

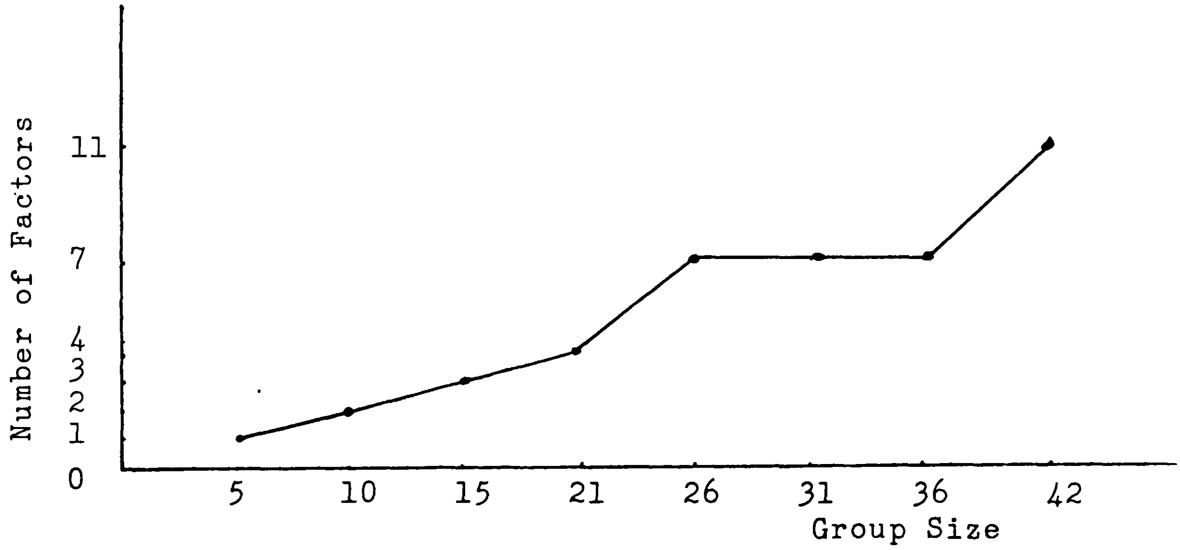


Eleventh master group and its subgroups.

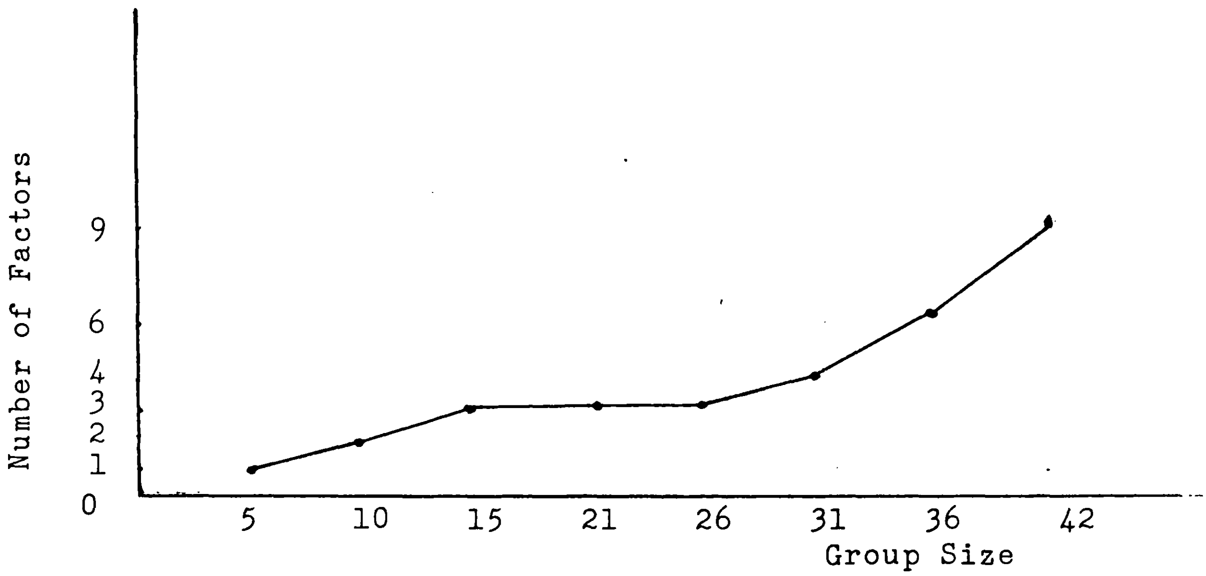


Twelfth master group and its subgroups.

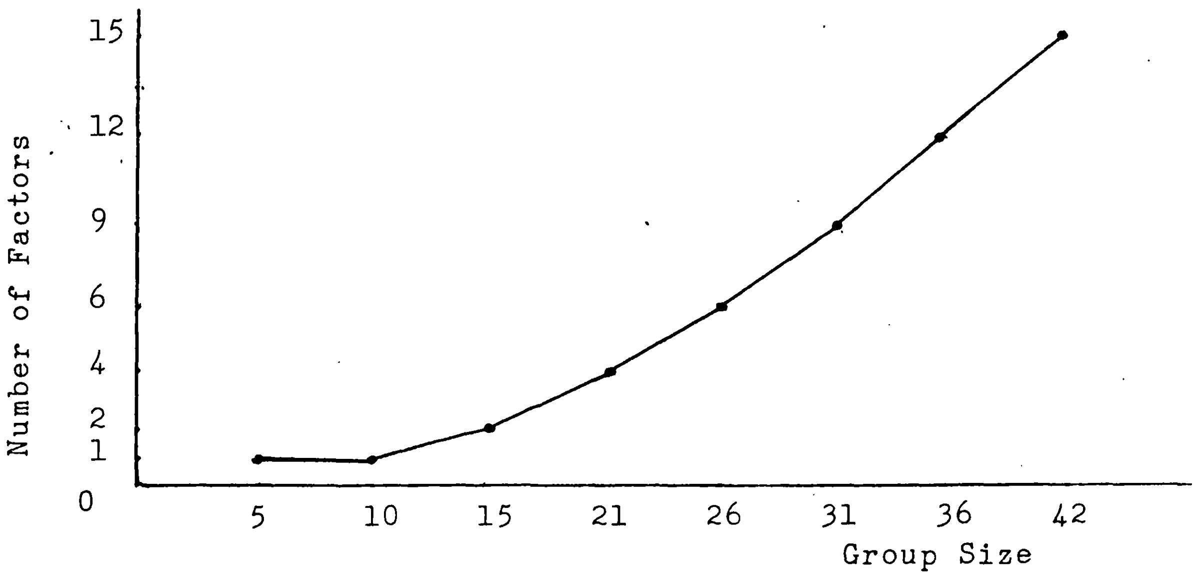
Figure 10.1
(Continued)



Thirteenth master group and its subgroups.

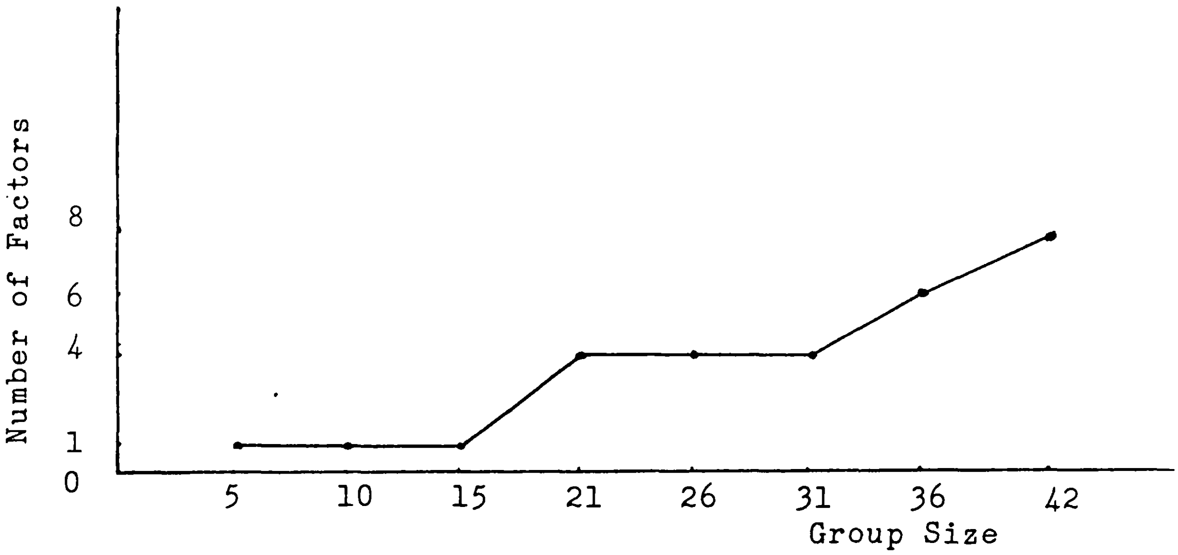


Fourteenth master group and its subgroups.



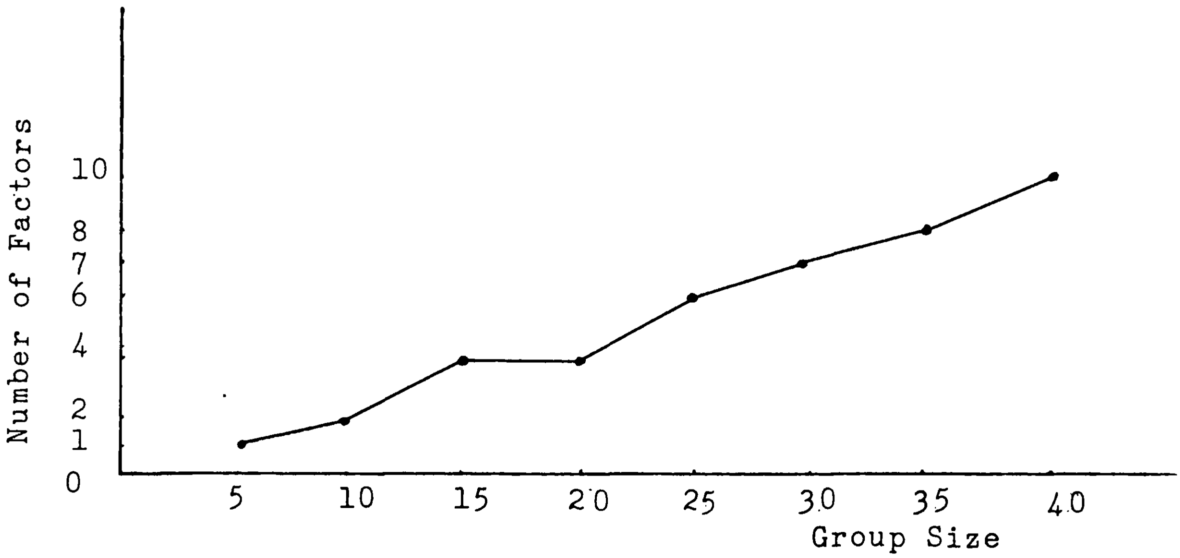
Fifteen master group and its subgroups.

Figure 10.1
(Continued)

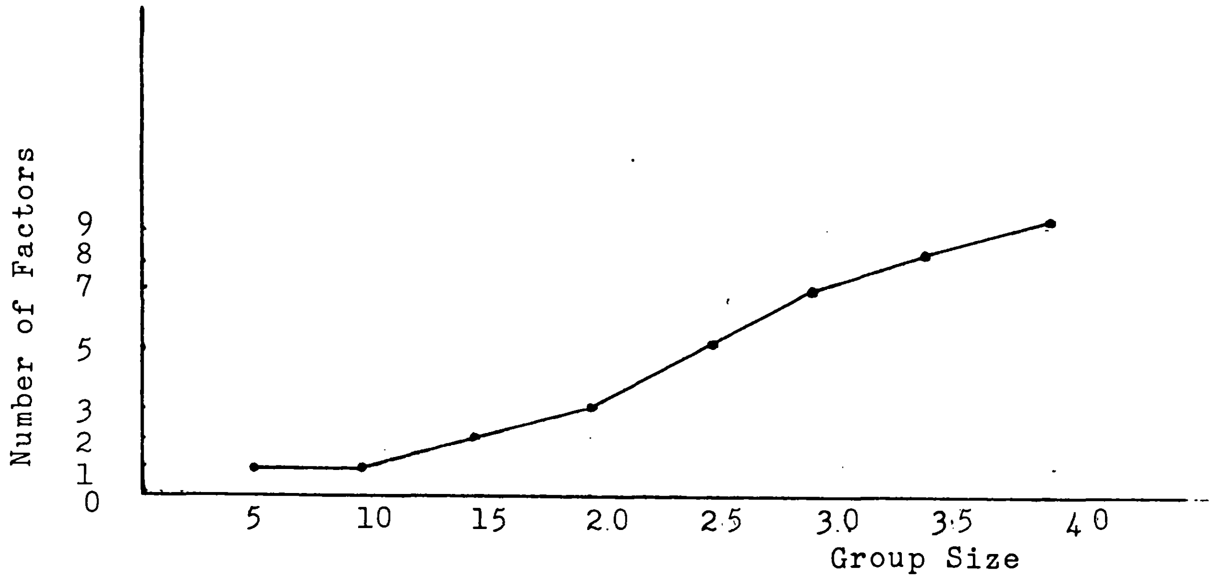


Sixteenth master group and its subgroups.

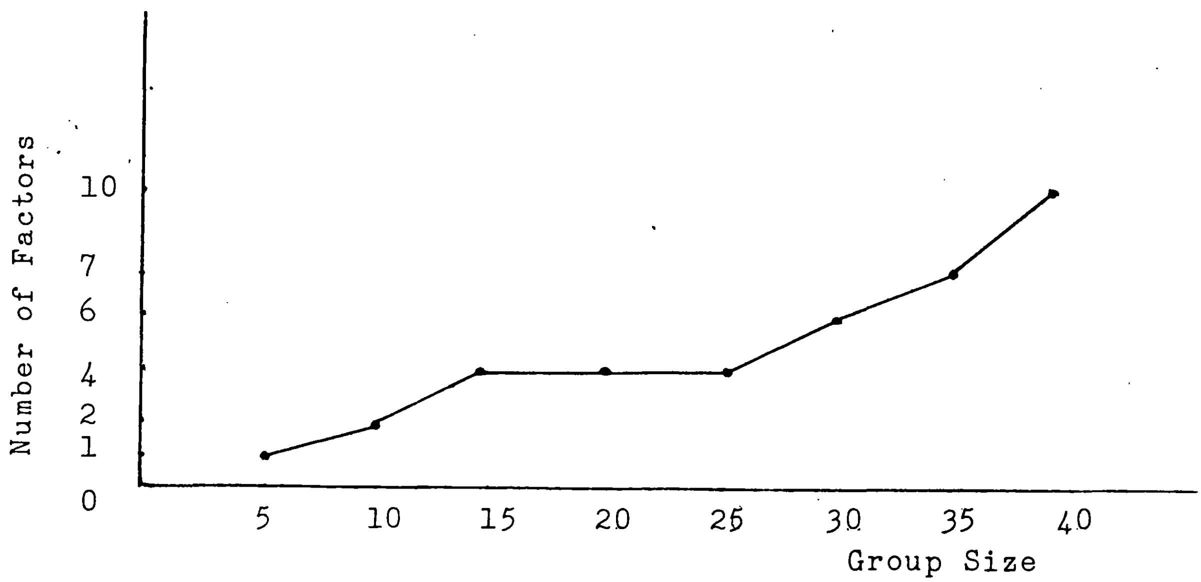
Figure 10.2 Graphical representation between the number of factors and the group size.
Sample B: Period:11/1956-12/1981,
Number of securities:200 .



First master group and its subgroups.

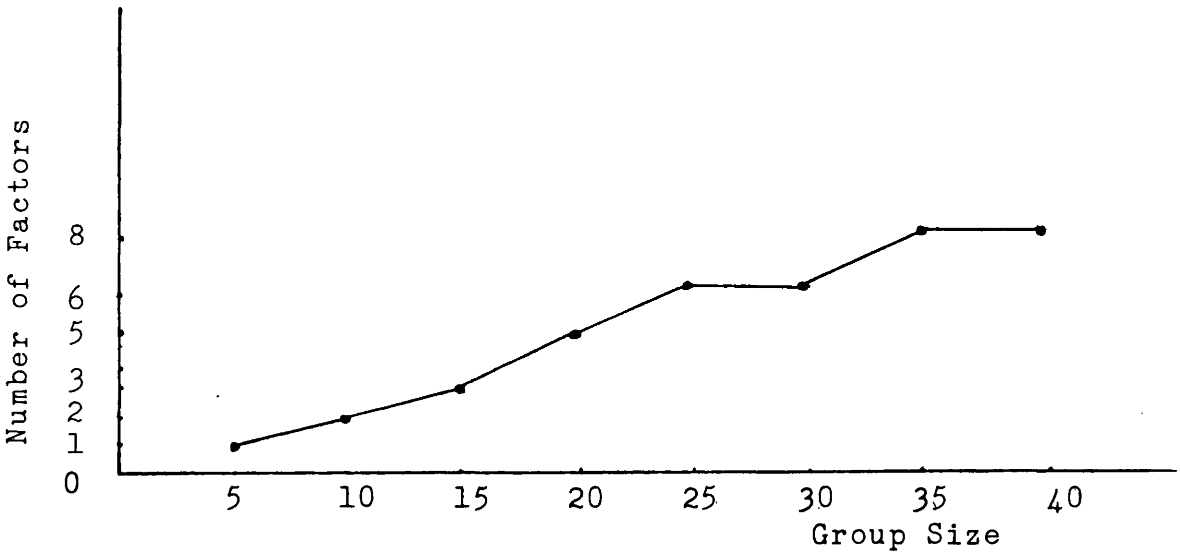


Second master group and its subgroups.

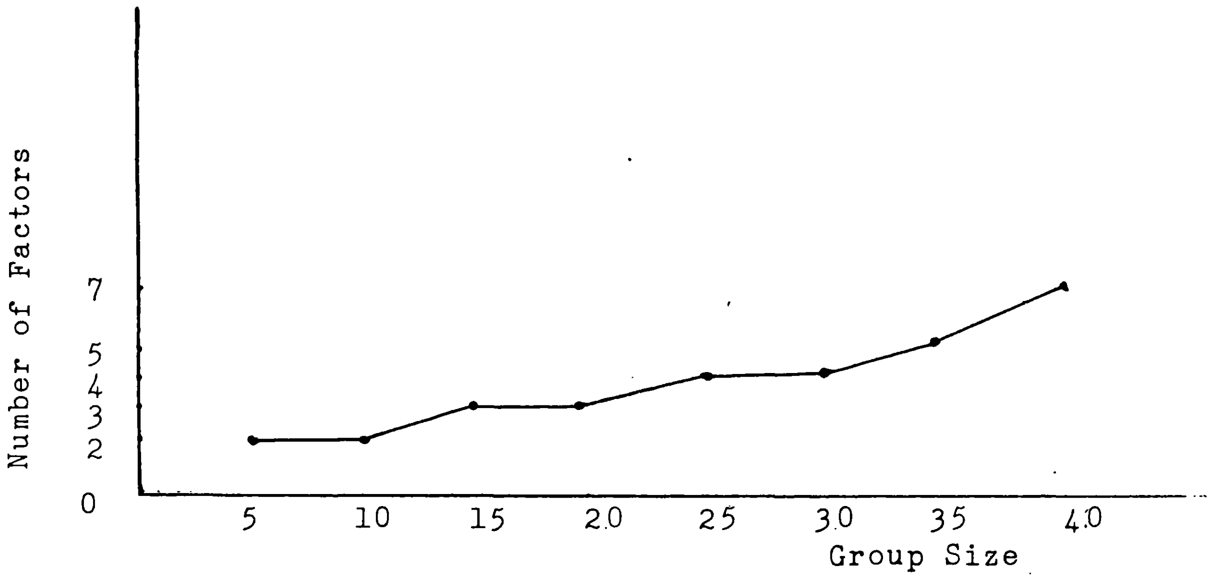


Third master group and its subgroups.

Figure 10.2
(Continued)



Fourth master group and its subgroups.



fifth master group and its subgroups.

Consequently by making use of sample A it can be inferred that the number of factors having influence on security returns is not the same across various security groups of different sizes.

For the groups of sample A the number of factors is ranged from 1 to 16. Moreover if the number of factors is considered as a function of the group size, then there is not an explicit mathematical formula of such a function.

Table 10.3 also provides some evidence of the number of factors yielding from security groups of the same size. Such evidence shows that, in general, the number of factors is not the same across various groups of the same size. For example compare the first group of 31 securities yielding 7 factors with the fourth group of 31 securities yielding 12 factors. Furthermore, if one compares the number of factors emerged from groups of the same size, it can be found that the higher variability concerning the number of factors occurs when groups of size 36 are factor analyzed.

The results of Table 10.3 also indicate that the number of factors changes as the group size changes, even if groups contain no common securities. In this case the number of factors do not always increase with the group size. For example, compare the first group containing 15 securities and emerging 6 factors with the third group containing 21 securities and emerging 3 factors.

In Table 10.3 are also presented for each security group the commulative percentage of total variance accounted for by the common factors. In most of the cases the commulative percentage of total variance accounted for by the common factors increases as the group size increases. The higher increase occurs when the master group 4 is considered. In this case the group of size 5 yields one factor and the proportion of the total variance accounted for by this common factor is 49.9 , while the group of size 42 yields 16 factors and the proportion of total variance accounted for by these common factors is 85.9 .
For each group of securities of sample A the first factor explains the larger proportion of the variance accounted for by all the relevant common factors. Table 10.5 gives an example indicating the importance of the first common factor. The results show that the proportion of total variance accounted for by the first factor decreases as the group size increases. Moreover the proportion of total variance accounted for by each of the remaining common factors decreases as the group size increases. However, the decrease in the former case is greater than the decrease in the latter cases.
This in turn explains the positive relationship between the number of factors and the group size.

Next as shown in Table 10.4 if one uses sample B and considers each master group with its subgroups, the number of factors changes as the group size changes in 93 cases out of 100 (i.e. 130 cases out of 140). In such cases the number of factors increases as the group size increases. By averaging the number of factors across security groups of

Table 10.5 An example indicating the importance of the first factor.
Sample A: Period: 1/1972-12/1981 .

GROUP SIZE [1]	NUMBER OF COMMON FACTORS	PROPORTION OF TOTAL VARIANCE ACCOUNTED FOR BY THE FIRST COMMON FACTOR	PROPORTION OF TOTAL VARIANCE ACCOUNTED FOR BY EACH OF THE REMAINING COMMON FACTORS	COMMULATIVE PROPORTION OF TOTAL VARIANCE ACCOUNTED FOR BY THE REMAINING COMMON FACTORS
5	1	67.5	0	0
10	3	61.2	7.6,6.4	14
15	6	52.1	7.4,6.2,5.5,5.1,4.4	28.6
21	6	50.2	6.7,5.4,4.5,4.2,3.9	24.6
26	7	50.9	5.9;4.4,4.0,3.7,3.4, 3.3	24.7
31	7	49.9	5.1, 4.0,3.7,3.6,3.1, 2.9	22.5
36	10	49.8	4.6,3.6,3.5,3.3,3.0, 2.6,2.5,2.3,2.1	27.4
42	13	48.9	4.4,3.4,3.3,2.9,2.7, 2.4,2.3,2.1,1.9	24.6

1 The master group used in this example is the first out of the 16 master groups of sample A .

the same size it is clear that the number of factors increases with the group size.
Therefore the utilization of sample B shows that the number of factors which determine the security returns, is not the same across various security groups of different sizes.

For the groups of sample B the number of factors is in the range 1 to 10. Furthermore there is not a specific mathematical formula to represent the relationship between the number of factors and the group size.

The empirical findings of Table 10.4 also indicate that the number of factors is not the same across various groups of the same size. As examples, the first group of 30 securities yields 7 factors while the fifth group of 30 securities yields 4 factors. The higher variability regarding the number of factors is obtained when groups of size 35 or groups of size 40 are factor analyzed.

Table 10.4 also compares the number of factors and the size of groups which contain no common securities. Such comparison shows that the number of factors changes as the group size changes, but the number of factors does always increase with the group size. As examples, from the first group of 25 securities 6 factors are emerged, whereas from the fifth group of 30 securities 4 factors are emerged.

The results of Table 10.4, concerning the relationship between the number of factors affecting the security returns and the group size, are in line with the results of Table 10.3. However, the number of factors determining the security returns of sample B is smaller than the number of factors affecting the security returns of sample A. For example the sixth group of 36 securities yields 12 factors (see Table 10.3), while the third group of 35 securities yields 7 factors (see Table 10.4).

From Table 10.4 it is clear that in most of the cases the commulative percentage of total variance accounted for by the common factors increases as the group size increases. The higher increase can be observed when the master group four is considered. In such a case the group of size 5 yields one factor and the amount of the total variance accounted for by this common factor is 45.2, while the group of size 40 yields 10 factors and the amount of total variance accounted for by these common factors is 65.1.
For each security group of sample B the first factor is the most important and it explains the larger proportion of the variance accounted for by the relevant common factors. An example indicating the importance of the first factor is shown in Table 10.6.
From Table 10.6 several observations could be made. First, the importance of the first extracted factor decreases when moving from groups of size 5 to groups of size 10. Second, the importance of the remaining common factors decreases after group size increases. However, the decrease in the first case is greater than the decrease in the second case. This in turn justifies the existence of the positive relationship between the number of factors and the group size.

Table 10.6 An example indicating the importance of the first factor.
Sample B : Period : 11/1956- 12/1981 .

GROUP SIZE [1]	NUMBER OF COMMON FACTORS	PROPORTION OF TOTAL VARIANCE ACCOUNTED FOR BY THE FIRST COMMON FACTOR	PROPORTION OF TOTAL VARIANCE ACCOUNTED FOR BY EACH OF THE REMAINING COMMON FACTORS	COMMULATIVE PROPORTION OF TOTAL VARIANCE ACCOUNTED FOR BY THE REMAINING COMMON FACTORS
5	1	54.6	0	0
10	2	51.9	10.3	10.3
15	4	49.3	7.8,6.0,5.4	19.1
20	4	46.7	5.9,4.9,4.7	15.5
25	6	47.2	5.0,4.2,3.9,3.5,3.3	19.9
30	7	47.3	4.8,3.6,3.5,3.4,2.9, 2.8	20.9
35	8	41.1	4.7,3.7,3.3,3.0,2.8, 2.7,2.7	22.6
40	10	39.8	4.2,3.4,3.1,2.9,2.6, 2.5,2.4,2.4,2.2	25.2

1 The master group used in this example is the first out of the 5 master groups of sample B .

Third, the results of Table 10.6 are similar to those of Table 10.5.

One possible way to obtain a clear description concerning the relationship between the average number of factors and the group size is to run a cross-sectional regression having as dependent variable the average number of factors and as independent variable the group size. For each sample the results of the cross-sectional regression are shown in Table 10.7. These findings reveal positive and significant relationships between the average number of factors and the group size. Moreover the results indicate that more than 94% of the variation in the average number of factors is explained by the group size.

Figures 10.3 and 10.4 show the least squares regression lines fitted to data on group size and average number of factors,using samples A and B respectively.

Next, a comparison of the total results produced by utilizing samples A and B gives :

(1) The invalidity of the assumption that the number of factors is the same across various groups was derived using sample A comprised of 128 groups and sample B comprised of 40 groups. Therefore by considering the number of groups of both samples the results concerning the present assumptions are more powerful and reliable.

Table 10.7 Cross-sectional regressions of the average number of factors on the group size.

	ALPHA COEFFICIENT[1]	BETA COEFFICIENT	ADJUSTED COEFFICIENT OF DETERMINATION
Sample A: Period: 1/1972-12/1981 Number of securities:672	-1.83[2] (-2.45*)[3]	0.32 (11.4)	94.8[4]
Sample B: Period: 10/1956-12/1981 Number of securities:200	-0.25 (-1.03*)	0.22 (22.5)	98.6

1 The cross-sectional regression equation is:

$$Y_u = a + bX_u + e_u$$

where
- $u=1,2,\ldots,8$.
- Y_u= the average number of factors.
- X_u= the group size.
- a,b= the regression coefficients.
- e_u= the error term.

2 The regression parameters a and b estimated by using the ordinary least squares method.

3 t-statistics appear in the parentheses. The null hypothesis is that the regression coefficient is equal to zero. Asterisks indicate that the null hypothesis is accepted at the 99% level of confidence.

4 The coefficient of determination is adjusted for 6 degrees of freedom.

Figure 10.3 Line fitted to data on group size and average number of factors. Sample A : Period : 1/1972 - 12/1981, Number of securities : 672.

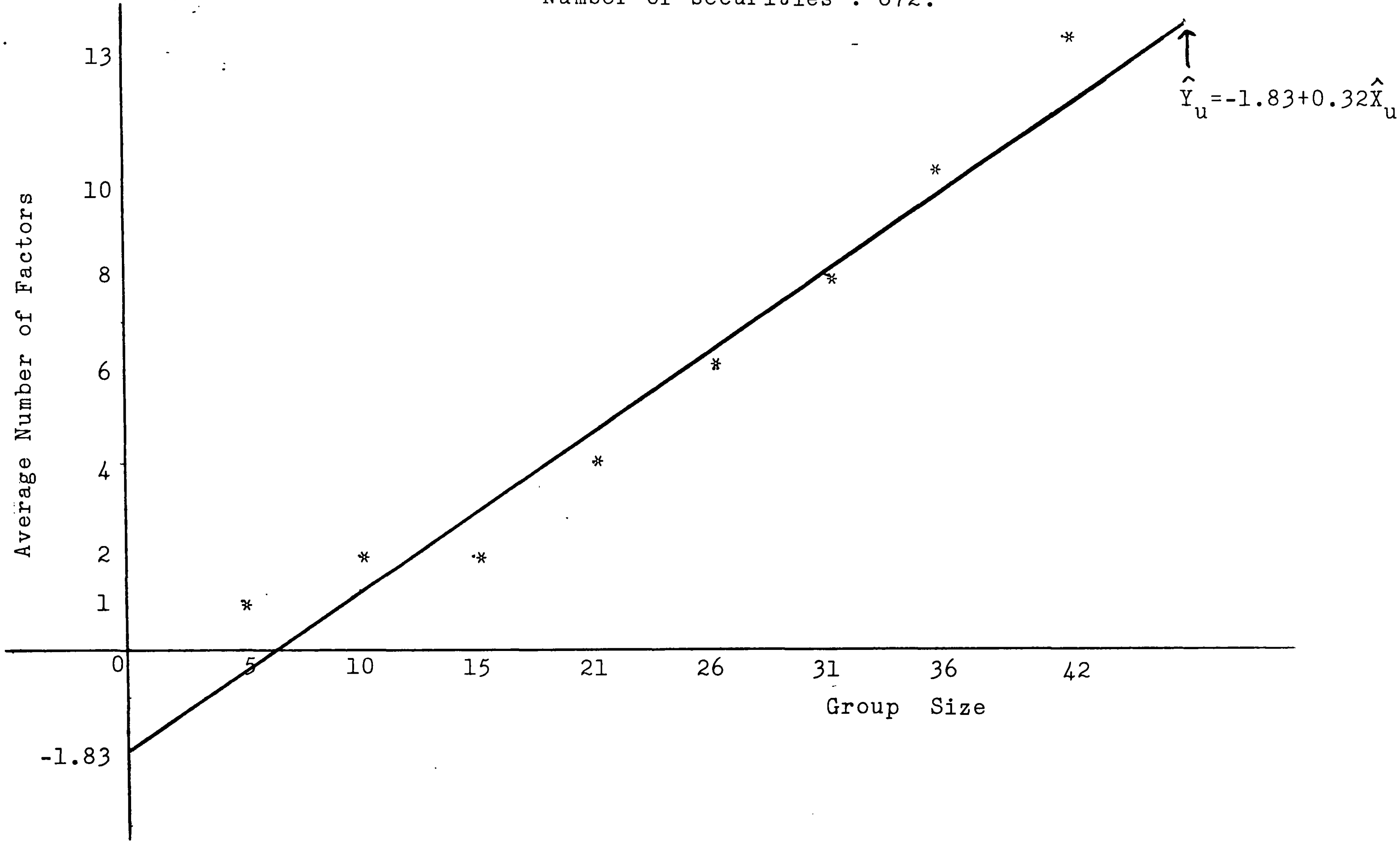
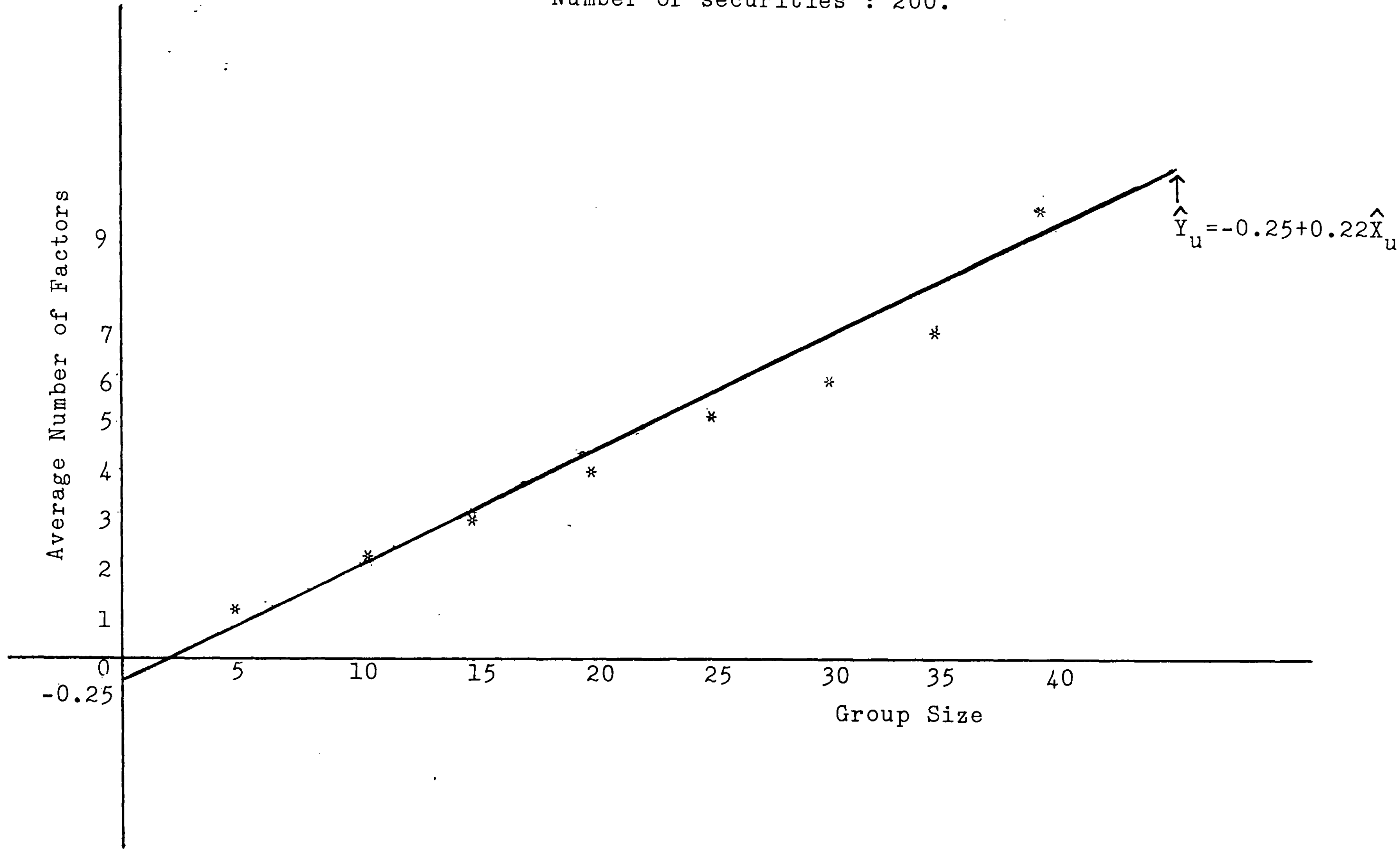


Figure 10.4 Line fitted to data on group size and average number of factors. Sample B : Period : 11/1956 - 12/1981, Number of securities : 200.



(2) The invalidity of the present assumption also occurs by considering sample B, which contains securities with observable returns over a long time period. Thus under these circumstances the invalidity of the present assumptions is not due to the consideration of a small number of observations per security.

Finally, summarizing the conclusions derived from the results of Tables 10.3 and 10.4, it can be stated :

(1) The number of factors affecting the security returns is not the same across various groups of different sizes.

(2) The number of factors determining the security returns is not the same across various groups of the same size.

10.3 Comparison with Previous Studies

There are only three previous studies which have investigated the relationship between the number of factors and the group size.

Kryzanowski and Chau(1982) used only one master group of securities and four (overlapping) subgroups and they concluded that the number of relevant factors is an increasing function of the size of the group being factored. They arrived at their conclusion by employing both Rao's and alpha factor analytic techniques. Moreover they found that both techniques produced the same number of factors.

Initially their conclusions can be criticized by the lack of statistical power. By considering only one group of securities does not mean that the other groups will produce the same results. The empirical findings of the present chapter indicate that the number of factors is positively related to the group size, but there are cases where the number of factors does not increase with the group size.

Also they concluded that the eigenvalue-one criterion of the alpha factor analysis gives the same results with the statistical test of Rao's factor analytic method. However, the eigenvalue-one rule of thumb is not a reliable criterion. In this study the eigenvalue-one criterion was violated in 90 cases out of 100. There were few cases where Rao's chi-square test produced relevant factors with eigenvalue greater than one, as well as cases where such a test produced relevant factors with eigenvalues less than one. For the second group of cases, there were factors producing significant chi-square values although their corresponding eigenvalues were in the range of .85 - .90 . Therefore it can be inferred that it seems dangerous to apply the eigenvalue-one criterion to choose the relevant number of factors.

Another empirical investigation of the relationship between the number of factors and the group size is offered by Dhrymes, Friend and Gultekin (1982). By utilizing one master group of securities and four (overlapping) subgroups, they deduced that the number of relevant factors is an increasing function of the group size being factored.

The results of Dhrymes, Friend and Gultekin, as those of Kryzanowski and Chau, reveal the lack of statistical power.

Kryzanowski and Chau used to test the goodness of fit of the factor model Rao's factor analytic technique ; while Dhrymes, Friend and Gultekin utilized for the same purpose the maximum likelihood factor analysis. It was pointed out,however, in Section 7.4 that both methods use different algorithms to estimate the factor loadings and factor scores, and both algorithms produce factor loadings and factor scores that constitute maximum likelihood estimates.
Therefore, by taking together the findings of Kryzanowski and Chau and Dhrymes, Friend and Gultekin it seems that both factor analytic techniques imply a positive relationship between the number of factors and the group size. However, this conclusion would be more reliable if Kryzanowski and Chau and Dhrymes, Friend and Gultekin used more master groups of securities.

Johnson (1981) also examined the relationship between the number of factors and the size of groups containing no common securities. Johnson's testing design is only a special case of the testing design employed in this study. According to his results he concluded :

> "There was no clear evidence to support the hypothesis that the number of factors is a function of the number of companies included in the groups. As examples, compare group 5 in sample 1 containing 52 companies and yielding 7 factors with group 1 in sample 5 having 15 companies and yielding 6 factors ". (p.38).

His conclusion that the number of factors changes across different non-overlapping security groups is similar with the conclusion on the same point of this study. But the number of factors that he found is less than the number of factors observed in this work. (He found that the number of factors is ranged from 3 to 11). Unfortunately Johnson did not state the method used to derive at these results. However, if he utilized the eigenvalue-one criterion his results are misleading, since it was explained previously that such a criterion is not reliable when U.K. data are utilized.

10.4 Some Possible Explanations of the Results

It was found that the number of factors affecting the security returns changes with the group size. One potential explanation of this result could be the violation of the normal distribution of security returns assumption. However, the results of the previous chapter show that this is not the case, because the joint distribution of security monthly returns can be approximate a multivariate normal

distribution.
Furthermore, if the joint distribution of security monthly returns deviates from a multivariate normal distribution, Joreskög (1963) concluded that the chi-square test for the goodness of fit of the factor model is robust to non-normality.

Another possible explanation of the results probably due to the limited number of security monthly observations for the London Stock Exchange and hence to the utilization of small sample sizes in terms of time periods. This may be true since the value obtained by equation (8.24) approximates a chi-square distribution only if the number of observations is large. A small number of observations may imply high correlated returns which in turn increase the value obtained by equation (8.24). Therefore more factors will be required to produce a válue that approximates a chi-square distribution. This point will be considered further in the next chapter.

Next a major disadvantage of the sequential procedure is that the critical value of the test criterion is fixed, while the null hypothesis of the number of factors is being tested in sequence and thus different chi-values are produced. As a consequence the number of factors will increase with the group size. If this is the case, factors will be emerged which represent only statistical artifacts and hence the produced results will be unrealistic. This in turn would indicate the inability of the factor analysis solutions to describe security returns generating models.

The possible explanations stated previously are concerned with the mathematical model's assumptions used to test the relationship between the number of factors and the group size. However, these explanations are not the only explanations of the results.

To test the assumption that the number of factors is the same across different groups two random samples of securities were utilized. These random samples generated a large number of random groups. Therefore some of the groups may contain securities of the same industry, while others may be comprised of securities from different industries. As a result the number of factors changes across different security groups of the same size. Moreover, if one adds in a group which contains securities of some particular industry, new securities belonging to other industries, the number of factors will increase.

Also there are factors which account for by a large proportion of the variability on some securities, but their influence on other securities is negligible. Changes in technology, political crises in some foreign countries, increase in borrowing rates are some examples of factors found to affect some security returns, while the influence of these factors on other securities is negligible. Since the securities of the groups in this study were chosen randomly, it is possible to find groups of securities whose return are not highly affected by those factors, whereas for other groups of securities such factors are important in determining the returns. As a result the number of factors changes across various groups of the same size and across various groups of different sizes.

Also, it can be found factors which influence the security returns over short time periods, while they are unimportant over long time periods. A political crisis, an oil crisis, a war scare, etc., are factors found to be important over short time periods, but unimportant over long time periods. This may be considered as a potential justification of the result that the number of factors which influence the security returns of sample B (302 observations per security) is smaller than the number of factors affecting the security returns of sample A (120 observations per security).

Finally, it is noted that a factor may be needed for the explanation of the random security returns, but not for the explanation of the expected return on securities. Therefore when there exists a large number of factors determining the security returns only few factors may be "priced".

Hughes (1982) found that there are more than 12 factors affecting the security returns for the Toronto Stock Exchange, but only three or four factors were "priced".

10.5 The Implications of the Empirical Results

The validity of the A.P.M. is relied upon a unique security returns generating model in the sense that the returns of a large number of securities are affected by a small number of relevant factors and each security return is determined by the same factors. Unfortunately, the theory behind the A.P.M. does not specify the number of the relevant factors which impact on security returns, as well as the identity of these factors. Hence the security returns generating model of the A.P.M. is an unobservable model. As a consequence, the empirical examination of the A.P.M. is performed by utilizing techniques depending only implicitly on the underlying factors.

Moreover for the A.P.M.'s test there exist computational restrictions with regard to the number of securities that can be handled at one time. Therefore it is necessary to split the securities of the sample into different groups and perform factor analytic techniques separately for each group.

In view of the results reported in this chapter it can be concluded that Rao's factor analytic technique produces for the London Stock Exchange, different returns generating models for security groups of different sizes as well as for security groups of the same size. It was explained in Section 10.4 that such results may be due either to Rao's factor analytic technique or to the existence of different factors affecting the returns on securities of the randomly chosen groups. In either cases the following problems can be seen :

(1) The identification of the unique security returns generating model of the A.P.M.

(2) The absence of an explicit description of the factors

produced by factor analyized various security groups.

(3) The existence of different security returns generating model emerged by factor analyzed various groups of securities of different sizes and various security groups of the same size.

According to these problems it can be inferred that:

(i) There is no way to ascertain which is the appropriate group size that has to be utilized in order to investigate empirical the validity of the A.P.M. By using security groups having a given size it cannot be ascerted that the producing security returns generating model is the unique model of the A.P.M., since if such a model exists it is unobservable.

(ii) The basic assumption of the A.P.M. concerning the uniqueness of the security returns generating model is violated. Thus the A.P.M. cannot be tested unambiguously using time series data from the London Stock Exchange. As a consequence one may challenge the introduction of the A.P.M. into the literature as a testible alternative to the C.A.P.M.

It is evident that these disturbing situations do not imply necessarily the invalidity of the A.P.M. They simply show our inability to provide a rigourous statistical methodology to test the model.

The conclusions derived in this section about the empirical tests of the A.P.M. are very similar to those of Roll's (1977) concerning the testability of the C.A.P.M. As it was notified in Section 3.3 Roll pointed out that the C.A.P.M. may be valid, but it cannot be tested unambiguously since there exists the market portfolio identification problem. Given a mean-standard deviation portfoloio there is not a method to assess whether it provides a good proxy of the market portfolio. The tests performed by utilizing a market proxy and employing the appropriate statistical techniques are not tests of the C.A.P.M. They are simply tests of the mean-standard deviation efficiency of the chosen market proxy.

Similarly in the case of the A.P.M. without a specific delineation of what the security returns generating model is, it seems that there exists also an identification problem. Furthermore the employment of factor analytic techniques produced a positive relationship between the number of factors and the group size. Hence given a pre-specified group size there is no way to ascertain whether the security return generating model produced via factor analytic techniques is the unique generating model of the A.P.M. As a result the tests performed by using such a generating model are not necessarily tests of the A.P.M.

In the mean-standard deviation theory each mean-standard deviation efficient portfolio produces a security return-risk linear relationship having the same form with the C.A.P.M., but it is not the C.A.P.M. Similar situations are obtained in the A.P.T., since from security groups of different sizes different security returns generating models are emerged; each security return generating model may produce a security return-risk linear relationship having the same form as the A.P.M., but such a relationship may not be the A.P.M.

The previously mentioned conclusions regarding the empirical examination of the A.P.M. are similar to the conclusions of Shanken (1982). Shanken considered two equivalent sets of securities in the sense that the portfolios emerging by combining the securities of the second set have equal rate of return with the securities of the first set. According to the A.P.M. such equivalent security sets should yield the same security returns generating model as well as the same security pricing relationship. However, Shanken proved theoretically that equivalent security sets yield different security returns generating models and hence different return-risk linear relationships.
In view of his theoretical findings and the identification problem in factor analysis he argued that the relevant security returns generating model is unobservable and his argument is similar to this of Roll's concerning the empirical examination of the C.A.P.M. According to Shanken :

"Roll argues that empirical investigations of the C.A.P.M. which use proxies for the true market portfolio are really tests of the mean-variance efficiency of those proxies, not tests of the C.A.P.M. The C.A.P.M. implies that a particular portfolio, the market portfolio, is efficient. The theory is not testable unless that portfolio is observable and used in the tests.

Similarly, it is argued here that factor-analytic empirical investigations of the A.P.T. are not necessarily tests of that theory. In the case of the A.P.T., we are confronted with the task of identifying the relevant factor structure, rather than the true market portfolio. Whereas we have a reasonably clear notion of what is meant by "the true market portfolio," it is not clear in what sense, if any, a uniquely "relevant factor structure" exists. We noted in Secion II that there are, in general, many factor structures corresponding to equivalent sets of securities. The A.P.T. does not appear to provide a criterion for singling out one structure as the "relevant" one". (p.p.1135-1136).

10.6 Conclusions

By utilizing Rao's factor analytic technique this chapter presented evidence indicating that the number of factors is not the same across various groups of the same size and across various groups of different sizes.

These findings reveal that the existing methodology for the A.P.M.'s tests is not the appropriate one, and previous tests of the A.P.M. are not necessarily tests of the model.

The A.P.M. may be held, but the existing statistical methodology does not insure unambiguous tests of the model for the London Stock Exchange.

CHAPTER 11

THE STABILITY OF THE NUMBER OF FACTORS ACROSS VARIOUS TIME PERIODS

This chapter empirically investigates whether the number of factors determining the security returns remains unchanged across various time periods for the same group of securities and across different time periods for different groups of securities.

The chapter begins with a presentation of the statistical procedure followed and it reports the empirical findings. Then it describes the empirical results and it discusses some possible explanations of the results. The chapter closes by presenting the implications of the empirical findings.

To verify empirically that the number of factors remains unchanged through time the tests 9 and 10 described in Table 8.6 are adopted. That is a test for the complete independence of the correlation matrix of security returns and Rao's test for the goodness of fit of the factor model. The first test is preconditioned for the second ; Thus the second test will be performed only if the correlation matrices contain significant non-diagonal entries.

The above mentioned tests are separately employed for each of the two samples (A and B) considered in this work.

From sample A of 672 securities, sixteen random master groups of size 42 are generated and from each master group two subgroups of 10 and 21 securities, respectively, are drawn. The first sample period of 120 monthly observations is divided into the following non-overlapping subperiods : January 1972 to December 1976 and January 1977 to December 1981.

The sample of 200 securities is broken down into five random master groups of size 40 and from each master group two subgroups are selected containing 10 and 20 securities, respectively. From the second sample period of the 302 monthly observations the following subperiods are generated : November 1956 - February 1965, March 1965 - June 1973, July 1973 - October 1981, November 1956 - May 1969 and June 1969 - December 1981.

The statistical testing procedure is the same as this of the previous chapter, so its description is omitted.

Next, Tables E.1,E.2 and E.3 in Appendix E, show that the chi-square values always exceed the corresponding critical test values. Therefore the correlations between security returns estimated over the subperiods 1/1972 - 12/1976, 1/1977 - 12/1981, 11/1976 - 2/1965, 3/1965 - 6/1973, 7/1973 - 10/1981, 11/1956 - 5/1969 and 6/1969 - 12/1981 are significantly different from zero.

As a consequence the correlation matrices are appropriate for factor analysis and thus one can proceed to examine empirically the stability of the number of factors through time.

Table 11.1 presents the number of factors affecting the returns of A's securities during the subperiods 1/1972 - 12/1976 and 1/1977 - 12/1981.

Table 11.2 contains information on the number of factors determining the returns of Sample B's securities during the subperiods 11/1956 - 2/1965, 3/1965 - 6/1973 and 7/1953 - 10/1981. Lastly, Table 11.3 gives the empirical evidence about the number of factors which influence the returns of sample B's securities during the subperiods 11/1956 - 5/1969 and 6/1969 - 12/1981.[1]

The empirical results of Tables 11.2 and 11.3 are further illustrated in Figures 11.1 snd 11.2 , respectively.

11.1 Description of the Empirical Results

The results of Table 11.1 indicate that, by taking into consideration sample A, the number of factors does not remain unchanged across the subperiods 1/1972-12/1976 and 1/1977 - 12/1981 ; for group size of 10 securities only 9 cases out of 16 are having the same number of factors,

1 A detailed representation of the findings of Tables 11.1,11.2 and 11.3 is given in Appendix E .

Table 11.1 Number of factors across two nonoverlapping subperiods for the same group of securities. Sample A:Period:1/1972-12/1981 , Number of securities : 672 .

	GROUP SIZE 10		GROUP SIZE 21	
SUBPERIOD	NUMBER OF FACTORS	COMMULATIVE PERCENTAGE OF TOTAL VARIANCE ACCOUNTED FOR BY THE COMMON FACTORS	NUMBER OF FACTORS	COMMULATIVE PERCENTAGE OF TOTAL VARIANCE ACCOUNTED FOR BY THE COMMON FACTORS
1/1972-12/1976	1	65.1	5	76.7
	2	73.1	5	77.1
	1	55.2	5	74.6
	1	51.1	8	83.4
	2	67.0	4	67.0
	1	57.4	8	84.4
	1	67.8	4	68.1
	1	49.5	5	72.6
	1	43.6	4	64.3
	1	44.6	3	59.7
	1	51.4	5	76.6
	2	70.1	5	76.4
	2	69.7	5	74.9
	2	71.3	4	74.2
	1	56.4	4	71.8
	1	49.6	5	69.4
1/1977-12/1981	2	62.5	3	57.5
	2	57.1	3	53.5
	1	42.4	3	52.2
	1	35.5	7	73.0
	1	44.1	3	53.3
	2	58.1	2	49.9
	1	68.2	3	56.2
	2	50.1	4	58.6
	1	37.4	3	53.2
	1	41.6	3	54.2
	1	42.7	3	52.3
	1	38.6	4	61.6
	1	41.6	3	52.3
	1	55.3	3	56.5
	1	52.4	4	64.4
	1	42.4	8	78.5

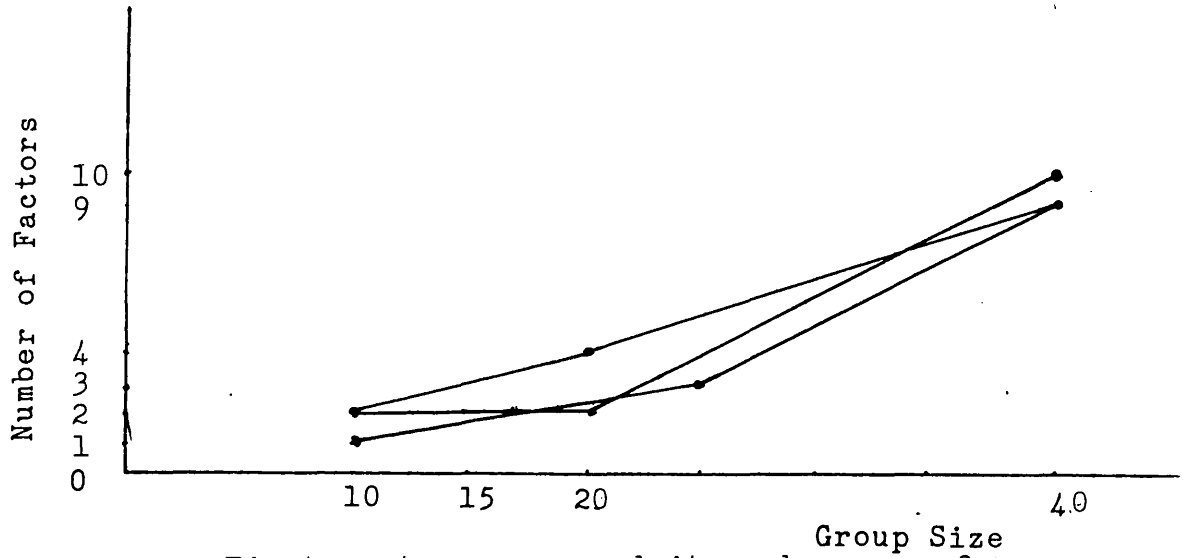
Table 11.2 Number of factors across three nonoverlapping subperiods for the same group of securities.
Sample B : Period :11/1956-12/1981 , Number of securities : 200 .

	GROUP SIZE 10		GROUP SIZE 20		GROUP SIZE 40	
SUBPERIOD	NUMBER OF FACTORS	COMMULATIVE PERCENTAGE OF TOTAL VARIANCE ACCOUNTED FOR BY THE COMMON FACTORS	NUMBER OF FACTORS	COMMULATIVE PERCENTAGE OF TOTAL VARIANCE ACCOUNTED FOR BY THE COMMON FACTORS	NUMBER OF FACTORS	COMMULATIVE PERCENTAGE OF TOTAL VARIANCE ACCOUNTED FOR BY THE COMMON FACTORS
11/1956-2/1965	1	47.8	3	51.8	9	66.0
	1	48.1	4	58.4	9	66.9
	1	44.0	2	43.3	7	58.5
	2	53.8	2	46.7	8	60.4
	2	52.6	2	40.5	7	56.8
3/1965-6/1973	2	57.8	2	47.0	10	69.9
	1	39.3	3	48.2	10	70.1
	2	61.2	3	51.0	9	63.9
	1	41.7	3	50.9	7	59.8
	1	40.4	2	45.3	6	55.2
7/1973-10/1981	2	67.9	4	69.2	9	76.3
	1	58.6	2	54.3	12	78.4
	2	70.3	4	67.0	10	75.9
	2	65.4	2	56.2	8	61.5
	2	64.9	3	62.4	7	60.3

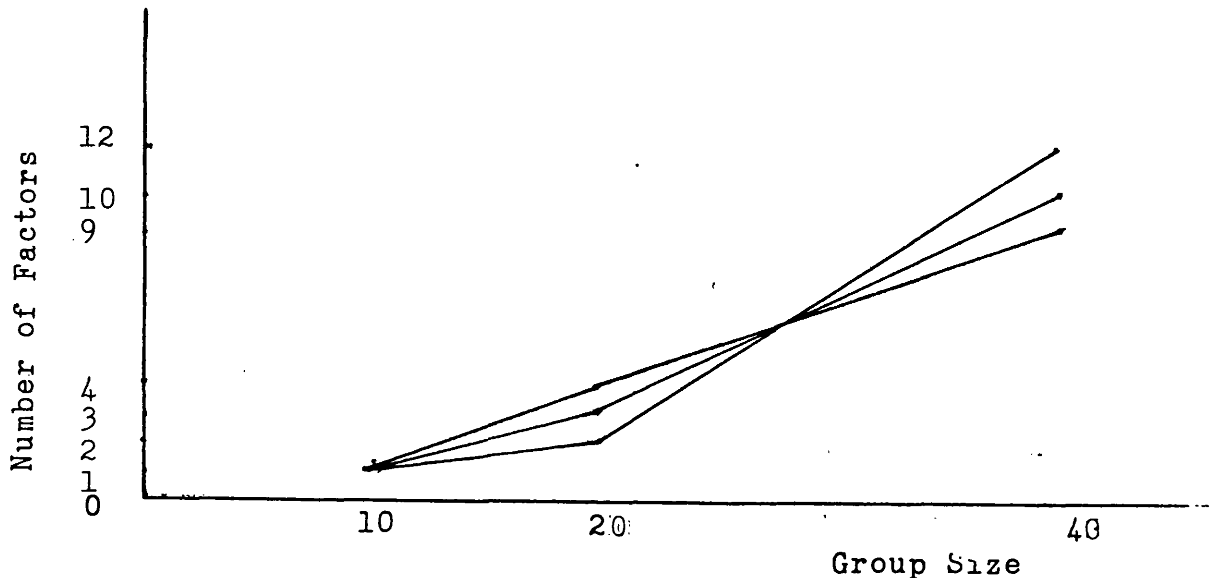
Table 11.3 Number of factors across two nonoverlapping subperiods for the same group of securities.
Sample : B : Period :11/1956-12/1981, Number of securities : 200.

	GROUP SIZE 10		GROUP SIZE 20		GROUP SIZE 40	
SUBPERIOD	NUMBER OF FACTORS	COMMULATIVE PERCENTAGE OF TOTAL VARIANCE ACCOUNTED FOR BY THE COMMON FACTORS	NUMBER OF FACTORS	COMMULATIVE PERCENTAGE OF TOTAL VARIANCE ACCOUNTED FOR BY THE COMMON FACTORS	NUMBER OF FACTORS	COMMULATIVE PERCENTAGE OF TOTAL VARIANCE ACCOUNTED FOR BY THE COMMON FACTORS
11/1956-5/1969	1	46.8	3	52.2	9	63.6
	1	37.6	5	65.9	7	52.5
	2	43.7	3	48.7	7	56.2
	1	36.0	4	58.6	6	53.6
	1	35.8	2	39.9	6	52.0
6/1969-12/1981	2	65.4	4	65.8	10	74.4
	1	53.5	4	62.3	9	72.2
	2	67.3	4	61.7	11	75.9
	1	54.2	2	52.8	8	67.3
	2	55.9	3	60.2	7	67.2

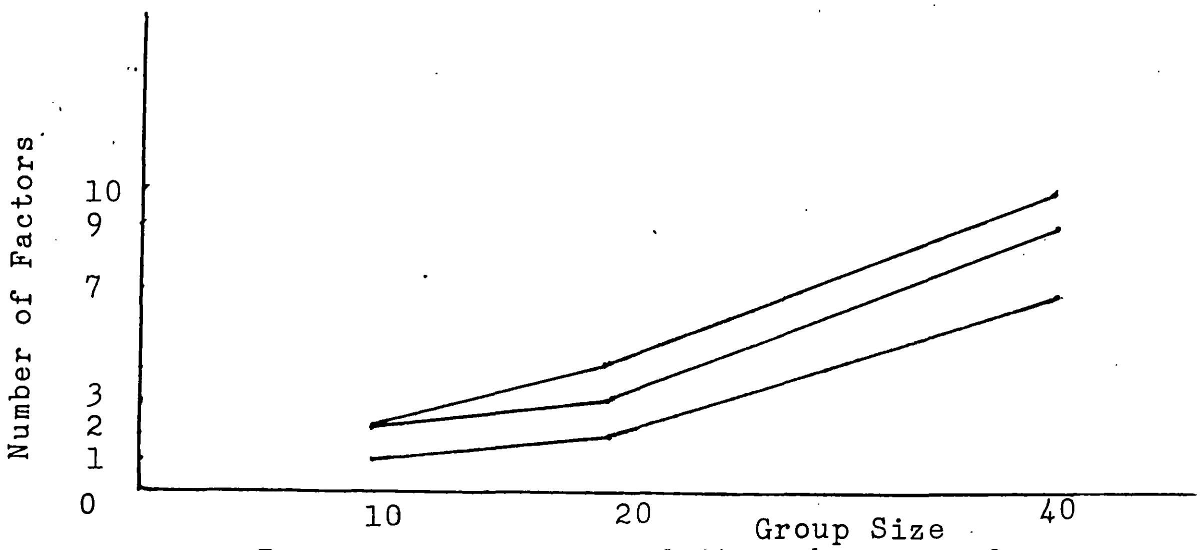
Figure 11.1 Graphical representation between the number of factors and the group size for three nonoverlapping subperiods. Sample B : Subperiods: 11/1956-2/1965, 3/1965-6/1973 and 7/1973-10/1981, Number of securities:200 .



First master group and its subgroups of sizes 10 and 20,respectively.

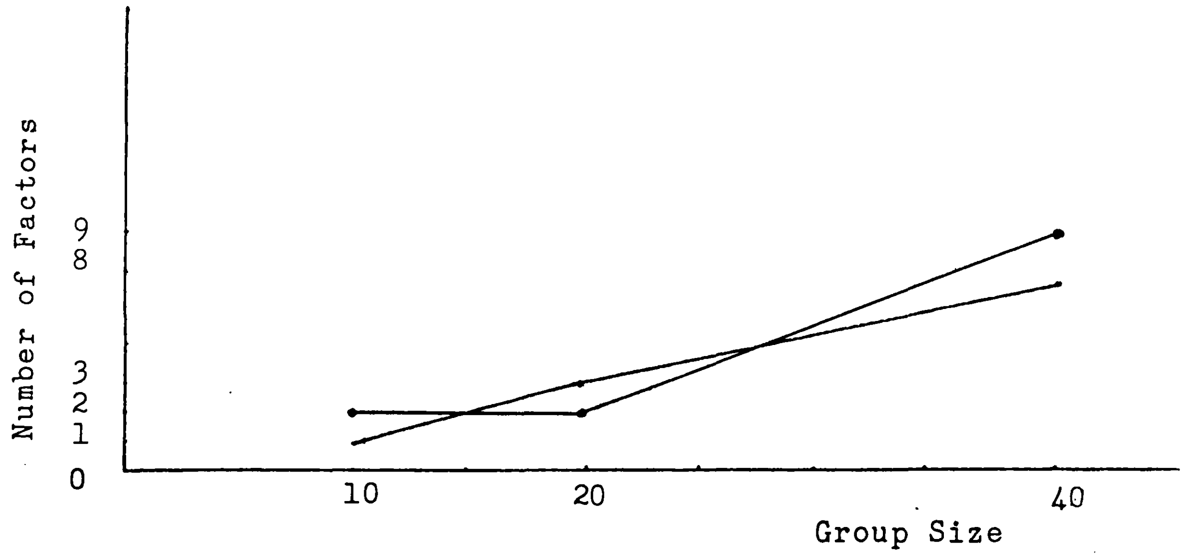


Second master group and its subgroups of sizes 10 and 20, respectively.

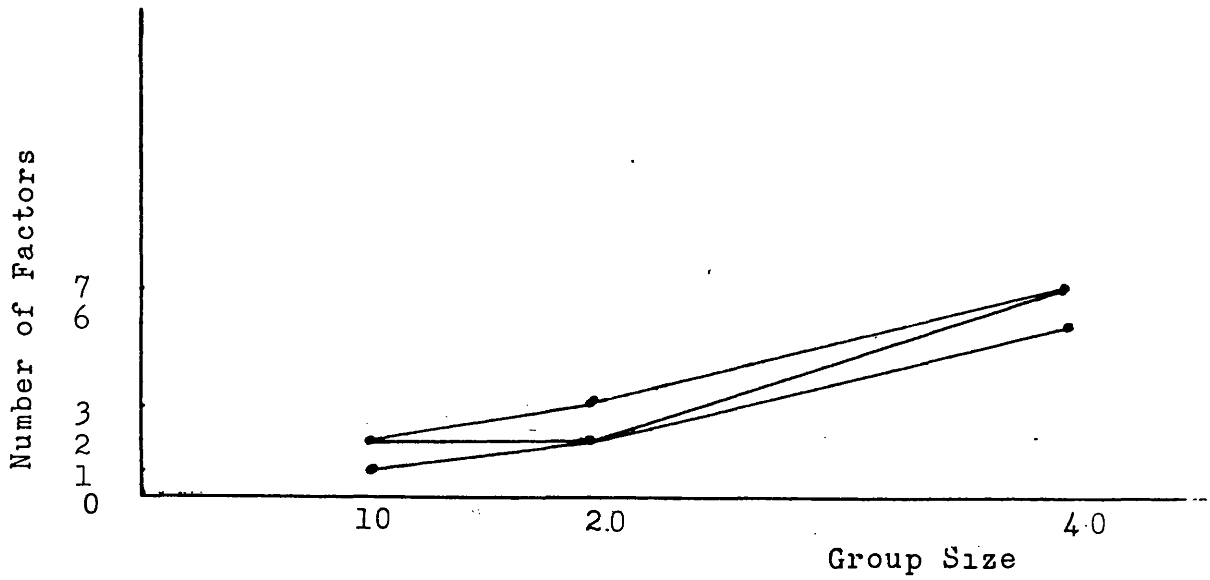


Third master group and its subgroups of sizes 10 and 20,respectively.

Figure 11.1
(Continued)

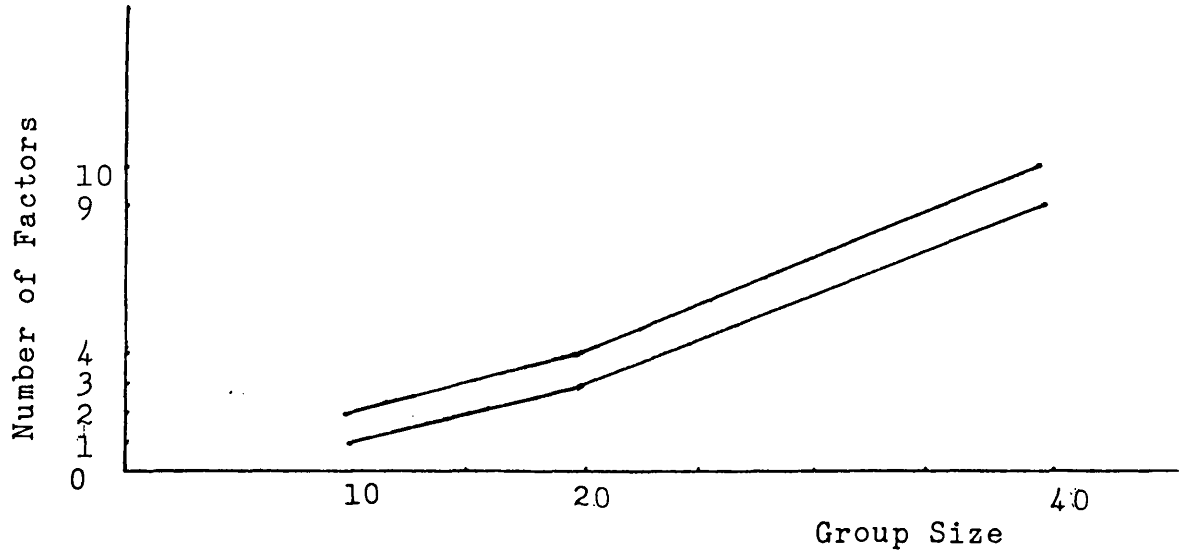


Fourth master group and its subgroups of
sizes 10 and 20 respectively.

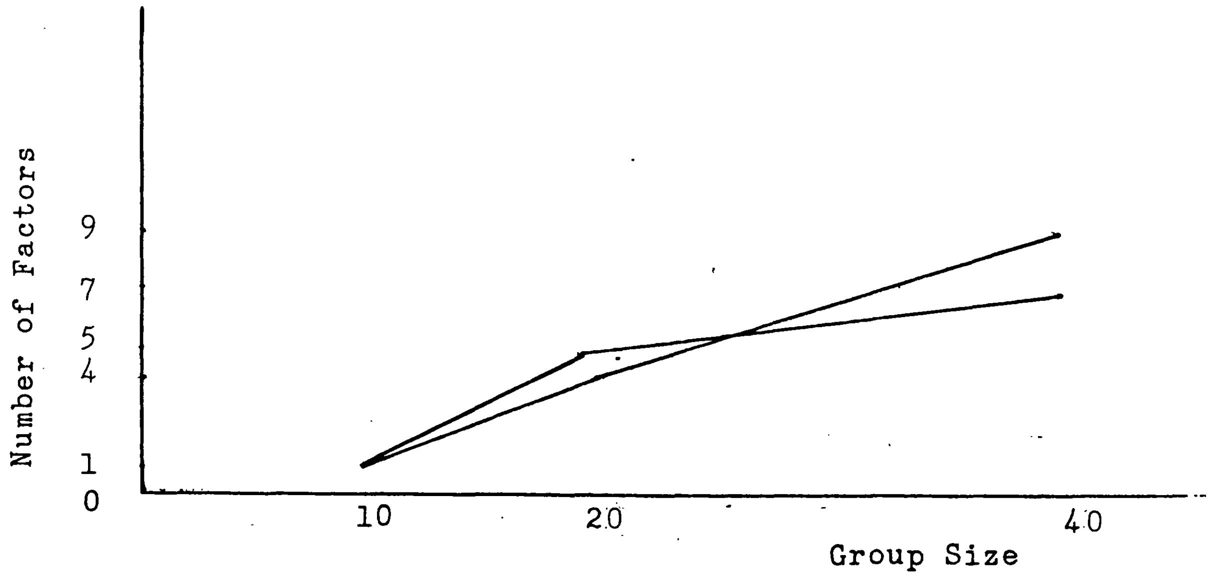


Fifth master group and its subgroups of
sizes 10 and 20 respectively.

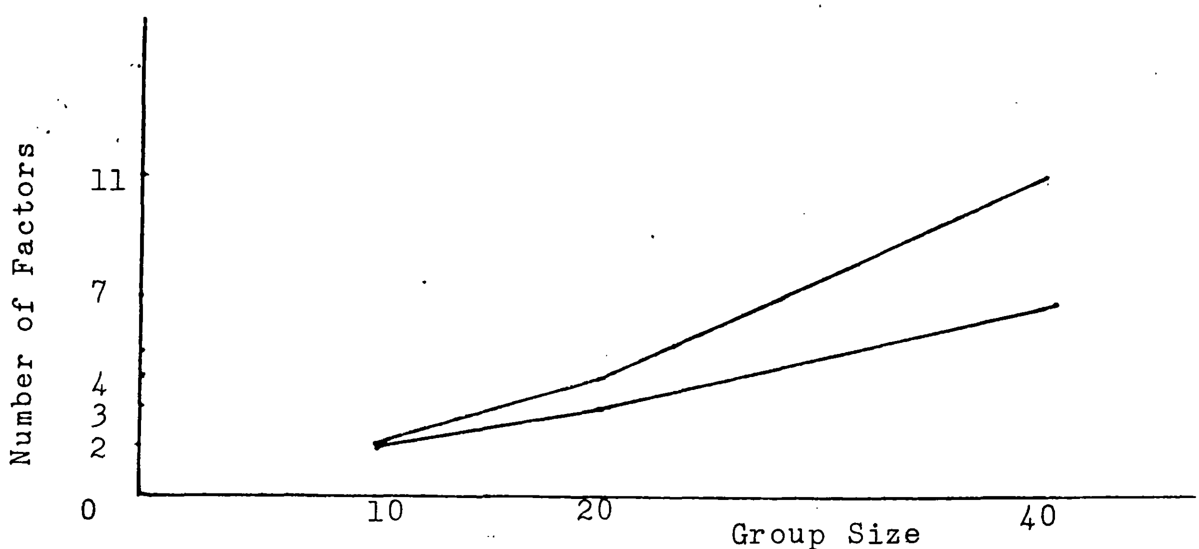
Figure 11.2 Graphical representation between the number of factors and the group size for two nonoverlapping subperiods. Sample B: Subperiods: 11/1956-5/1969 and 6/1969-12/1981,Number of securities: 200 .



First master group and its subgroups of sizes 10 and 20,respectively.

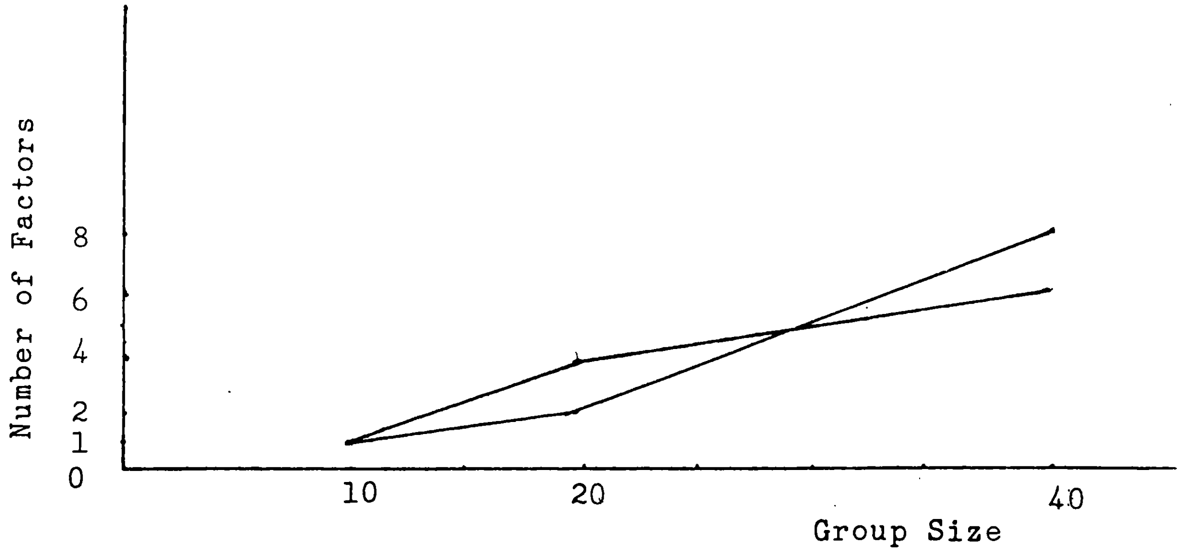


Second master group and its subgroups of sizes 10 and 20,respectively.

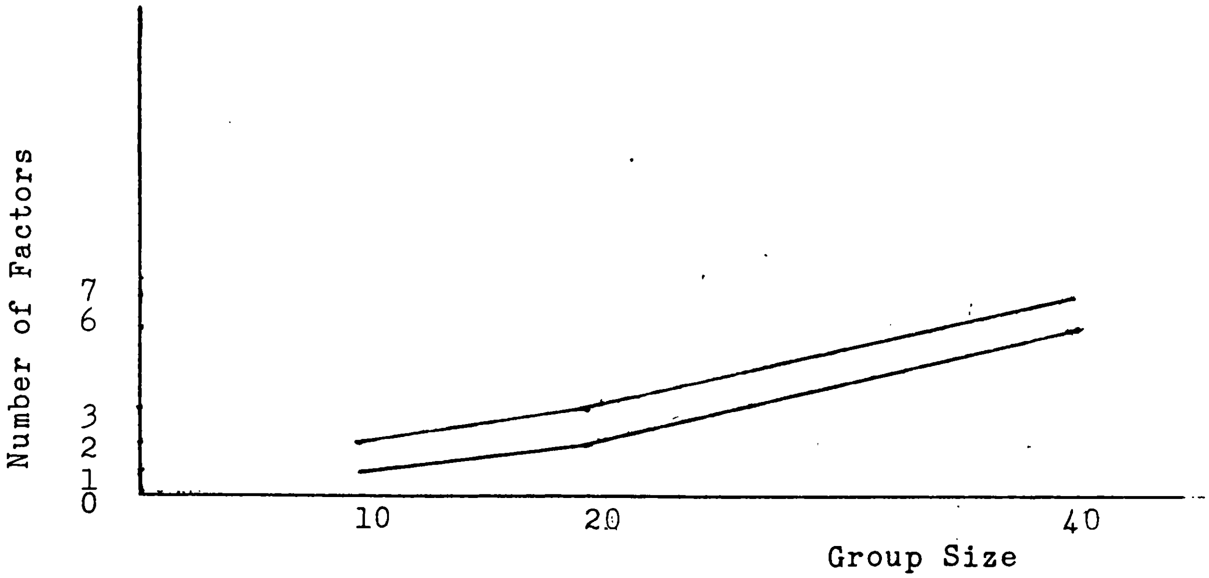


Third master group and its subgroups of sizes 10 and 20,respectively.

Figure 11.2
(Continued)



Fourth master group and its subgroups of size 10 and 20,respectively.



Fifth master group and its subgroups of sizes 10 and 20 respectively .

while for group size of 21 securities in only 2 cases out of 16 the number of factors remains unchanged .
As examples, for the first group of 21 securities required 5 factors to explain the variation in the returns during the subperiod 1/1972 - 12/1976, while for the same group only 3 factors are necessary to explain the validity in the returns during the subperiod 1/1977 - 12/1981.

It can also be seen that as the size of group increases the number of cases with the same factors across the subperiods decreases.
Therefore by using sample A it can be inferred that the number of factors does not remain unchanged across the two subperiods for the same group of securities .

Furthermore, from Table 11.1 it is clear that the number of factors is not the same across the subperiods 1/1972 - 12/1976 and 1/1977 - 12/1981 for various groups of securities of different sizes. As examples compare the first group containing 10 securities and yielding for the subperiod 1/1972 - 12/1976 one factor with the fourth group containing 21 securities and yielding for the subperiod 1/1977 - 12/1981 7 factors.

The results of Table 11.1 also reveal that the number of factors having influence on security returns does not remain unchanged across various security groups of different sizes and across various security groups of the same size. These results accord with the results of the previous chapter.

Hence they provide some additional evidence against the assumption that the same number of factors is emerged from various security groups of different sizes and from various security groups of the same size.

Comparing, however, the number of factors which affect the security returns during the total sample period 1/1972 - 12/1981 and the subperiods 1/1972 - 12/1976 and 1/1977 - 12/1981 it can be observed that there are cases where the relevant number of factors decreases as the number of observations increases. As examples, compare the fourth group containing 21 securities and emerging 5 factors responsible for the variation in the security returns during the entire sample period 1/1972 - 12/1981 (see Table 10.3), with the same group of securities which emerge 8 (7) factors responsible for the returns variability during the subperiod 1/1972 - 12/1976 (1/1977 - 12/1981).

In Table 11.1 are also reported for each security group the commulative percentage of total variance accounted for by the common factor. The results of Table 11.1 show for groups of size 10 the following :

(1) For the subperiod 1/1972 - 12/1976 the amount of the total variance accounted for by the common factors is ranged from 43.6 to 73.1.

(2) For the subperiod 1/1977 - 12/1981 the amount of the total variance accounted for by the common factors is ranged from 37.4 to 68.1 .

For groups of size 21 it can be asserted :

(1) For the subperiod 1/1972 - 12/1976 the proportion of the total variance accounted for by the common factors is in the range 59.7 to 84.4 .

(2) For the subperiod 1/1977 - 12/1981 the proportion of the total variance accounted for by the common factors is in the range 49.9 to 78.5 .

It is noticed from Table 11.1 that the percentages of total variation of returns which account for the common factors emerging during the subperiods 1/1972 - 12/1976 and 1/1977 - 12/1981 are different from the same group of securities.

For each security group of sample A the first extracted factor in the subperiods 1/1972 - 12/1976 and 1/1977 - 12/1981 is the most important and it explains a large portion of the total security variance. Table 11.4 presents an example indicating the importance of the first factor in each subperiod. The findings show that for each group size the first extracted factor during the subperiod 1/1972 - 12/1976 explains a larger portion of the security variance than it explains during the subperiod 1/1977 - 12/1981.
Referring again to Table 11.4, each of the remaining common factors during the first subperiod explains a smaller portion of the total security variance than it explains during the second subperiod. It can also be seen from Table 11.4 that the proportion of total security variance which can be explained by the first factor decreases as the group size increases. Furthermore

Table 11.4 An example indicating the importance of the first factor across two nonoverlapping subperiods. Sample A : Period : 1/1972-12/1981.

SUBPERIOD	GROUP SIZE [1]	NUMBER OF COMMON FACTORS	PROPORTION OF TOTAL VARIANCE ACCOUNTED FOR BY THE FIRST COMMON FACTOR	PROPORTION OF TOTAL VARIANCE ACCOUNTED FOR BY EACH OF THE REMAINING COMMON FACTORS	COMMULATIVE PROPORTION OF TOTAL VARIANCE ACCOUNTED FOR BY THE REMAINING COMMON FACTORS
1/1972-12/1976	5	1	72.5	0	0
	10	1	65.1	0	0
	15	3	55.7	8.2,6.4	14.6
	21	5	54.8	7.6,5.7,4.6,4.1	22.0
1/1976-12/1981	5	2	55.7	13.0	13.0
	10	2	51.9	10.6	10.6
	15	3	45.0	8.4,7.5	15.9
	21	3	41.3	9.4,6.8	16.2

1 The subgroups used in this example are the subgroups of the first master group of sample A .

the proportion of total variance which can be explained by each of the remaining common factors decreases as the group size increases but the decrease in this case is smaller than the decrease in the previous case.

These results justify the positive relationship between the number of factors and the group size and hence they are consistent with the result of Tables 10.5 and 10.6.

Next, in view of the findings in Table 11.2 it can be observed that by utilizing sample B, the number of factors changes across the subperiods 11/1956 - 2/1965, 3/1965 - 6/1973 and 7/1973 - 10/1981 ; for groups containing 10 securities only one case out of 16 is having the same number of factors, while for groups containing 40 and 20 securities there is not a single case which producas the same number of factors. As examples for the third group of 40 securities 7 factors are necessary to explain the variation in the returns during the subperiod 11/1956 - 2/1965, 9 factors are needed to explain the variability in the returns during the subperiod 3/1965 - 6/1973 and 10 factors are required to explain the variation in the returns during the subperiod 7/1973 - 10/1981.

Consequently on the basis of these results it can be deduced that the number of factors does not remain unchanged across various subperiods for the same group of securities .

Another conclusion derived from the results of Table 11.2 is that the number of factors changes across the three subperiods as the group size changes. For example the first group containing 20 securities yields for the subperiod 11/1956 - 2/1965 3 factors, whereas the first group containing 40 securities yields for the subperiod 3/1965 - 6/1973 10 factors.

In addition the results of Table 11.2 confirm that in almost all the cases the number of factors which influence the security returns changes across various security groups of different sizes and across various security groups of the same size for the same subperiod. These results are similar to the results of the previous chapter.

There are, however, some cases where the relevant number of factors decreases as the observations per security increase . For example, compare the second group containing 40 securities and yielding 9 factors affecting the security returns during the period 11/1956 - 12/1981 (see Table 10.4) with the same group of securities yielding 12 factors which influence the security returns during the subperiod 7/1973 - 10/1981.
The results of Table 11.2 are in line with the corresponding results of Table 11.1.

Table 11.2 also reveals for groups of size 10 the following :

(1) For the subperiod 11/1956 - 2/1965 the proportion of total variance accounted for by the common factors is ranged from 44.0 to 53.8 .

(2) For the subperiod 3/1965 - 6/1973 the proportion of total variance accounted for by the common factors is ranged from 39.3 to 61.7 .

(3) For the subperiod 7/1973 - 10/1981 the proportion of total variance accounted for by the common factors is ranged from 58.6 to 70.3 .

For groups of size 20 once can state the following :

(1) For the subperiod 11/1956 - 2/1965 the amount of total variance accounted for by the common factors is in the range 40.5 to 58.4 .

(2) For the subperiod 3/1965 - 6/1973 the amount of total variance accounted for by the common factors is in the range 45.3 to 50.9 .

(3) For the subperiod 7/1973 - 10/1981 the amount of total variance accounted for by lthe common factors is in the range 54.3 to 69.2 .

Lastly, for groups of size 40 it can be observed,the following :

(1) For the subperiod 11/1956 - 2/1965 the proportion of total variance accounted for by the common factors is ranged from 56.8 to 66.9 .

(2) For the subperiod 3/1956 - 6/1973 the proportion of total variance accounted for by the common factors is ranged from 55.2 to 70.1 .

(3) For the subperiod 7/1973 - 10/1981 the proportion of total variance accounted for by the common factors is ranged from 60.3 to 78.4 .

From these results it can be seen that the precentages of the total variability in returns that account for the common factors affecting the security returns during the three subperiods are different for the same group of securities. These results are in accordance with the corresponding results derived by utilizing the first sample.

The first extracted factor for each security group of sample B and in each subperiod is the most important and it explains a large proportion of the total security variance. An example indicating the importance of the first factor is shown in Table 11.5 . For each group size the proportion of total variance which can be explained by the first factor during the subperiod 7/1973 - 10/1981 is larger than the proportion of total variance accounted for by the first factor during the subperiods 11/1956 - 2/1965 and 3/1965 - 6/1973.
In addition, each of the remaining common factors during the first and second subperiods explains a larger proportion of the total security variance than it explains during the third subperiod. The results displayed in Table 11.5 also show that the proportion of total security variance accounted for by the first factor decreases as the group size increases. Moreover the proportion of the total security variance that can be accounted for by each of the remaining common factors decreases as the group size increases . However, the decrease in the latter case

Table 11.5 An example indicating the importance of the first factor across three nonoverlapping subperiods.
Sample B: Period:11/1956-12/1981 .

SUBPERIOD	GROUP SIZE [1]	NUMBER OF COMMON FACTORS	PROPORTION OF TOTAL VARIANCE ACCOUNTED FOR BY THE FIRST COMMON FACTOR	PROPORTION OF TOTAL VARIANCE ACCOUNTED FOR BY THE REMAINING COMMON FACTORS	COMMULATIVE PROPORTION OF TOTAL VARIANCE ACCOUNTED FOR BY THE REMAINING COMMON FACTORS
11/1956-2/1965	5	1	52.2	0	0
	10	1	47.8	0	0
	15	2	44.7	8	8
	20	3	40.1	6.0,5.7	11.7
	40	9	38.2	5.4,3.9,3.6,3.4,3.2,3.0,2.8,2.6	27.8
3/1965-6/1973	5	1	52.4	0	0
	10	2	45.1	12.7	12.7
	15	2	40.2	11.1	11.1
	20	2	38.4	8.6	8.6
	40	10	37.4	5.7,4.6,4.0,3.7,3.3,3.2,2.9,2.8,2.5	32.5
7/1973-10/1981	5	1	57.8	0	0
	10	2	57.4	10.5	10.5
	15	2	56.3	7.1	7.1
	20	4	54.1	5.7,4.7,4.6	15.1
	40	9	52.1	4.7,3.7,3.2,2.3,2.7,2.5,2.3,2.1	24.2

1 The master group used in this example is the first out of 5 master groups of sample B .

is smaller than the decrease in the former case.

With the aid of these results it can be justified the positive relationship between the number of factors the group size and hence to conclude the similarity of the results with those presented in Tables 10.5, 10.6 and 11.4 .

Next the results given in Table 11.3 show that by considering sample B, the number of factors does not remain unchanged across the subperiods 11/1956-5/1969 and 6/1969 - 12/1981 ; for group size of 10 securities only 3 cases out of 5 are having the same number of factors, while for groups containing 20 and 40 securities there is not a single case in which the same number of factors is produced. For example the third group of size 40 yields 7 factors affecting the security returns during the first subperiod whereas the same group yields 11 common factors determining the security returns during the second subperiod.

According to these results it can also be concluded that the number of factors does not remain unchanged across the two subperiods for the same group of securities.

The results of Table 11.3 also indicate that the number of factors changes across the two subperiods for various groups of securities of different sizes. For example the first group containing 20 securities yields for the subperiod 11/1956 - 5/1969 3 factors, while the first group containing 40 securities yields for the subperiod 6/1969 - 12/1981 10 factors.

The findings illustrated in Table 11.3 are against the validity of the assumption that the number of factors affecting the security returns remains unchanged across various security groups of different sizes and across various security groups of the same size. These findings are consistent with the results of the previous chapter.

A comparison of the number of factors determining the security returns during the total sample period 11/1956 - 12/1981 and the subperiods 11/1956 - 5/1969 and 6/1969 - 12/1981 also shows that there exist cases where the relevant number of factors decreases as the number of observations decrease. For example, compare the second group containing 20 securities and yielding 3 factors affecting the security returns during the entire period 11/1956 - 12/1981 (see Table 10.4), with the same group of securities yielding 5 factors having influence on security returns during the subperiod 11/1956 - 5/1969.

Comparing the results of Tables 11.2 and 11.3 it can be also seen that for some cases the relevant number of factors decreases as the number of observations increase. As examples, compare the second group containing 40 securities and producing 9 factors which affect the security returns during the subperiod 11/1956 - 2/1965 with the same group of securities producing 7 factors affecting the security returns during the subperiod 11/1956 - 5/1969.

In addition Table 11.3 gives the following information for groups of size 10.

(1) For the subperiod 11/1956 - 5/1969 the percentage of total variance accounted for by the common factors is in the range 35.8 to 46.8 .

(2) For the subperiod 6/1969 - 12/1981 the percentage of total variance accounted for by the common factors is in the range 53.9 to 67.9 .

Also for groups of size 21 it can be stated the following :

(1) For the subperiod 11/1956 - 5/1969 the percentage of total variance accounted for by the common factors is ranged from 39.9 to 65.9 .

(2) For the subperiod 6/1969 - 12/1981 the percentage of total variance accounted for by the common factors is ranged from 52.8 to 65.8 .

Lastly for groups of size 40 it can be observed :

(1) For the subperiod 11/1956 - 5/1969 the percentage of total variance accounted for by the common factors is in the range 52.0 to 63.7 .

(2) For the subperiod 5/1956 - 12/1981 the percentage of total variance accounted for by the common factors is in the range 67.2 to 75.9 .

These results show that the percentages of total variation of returns which account for the common factors affecting the security returns during the subperiods 11/1956 - 5/1969 and 6/1969 - 12/1981 are different for the same group of securities.

Utilizing each security group of sample B it was found that the first extracted factor during the subperiods 11/1956 - 5/1969 and 6/1969 - 12/1981 is the most important factor and it explains a large proportion of the total security variance. Table 11.6 reports the results of an example indicating the importance of the first factor in each subperiod.
For each group size the first factor extracted in the subperiod 6/1969 - 12/1981 explains a larger proportion of the total security variance that it explains in the subperiod 11/1956 - 5/1969. Furthermore, each of the remaining common factors during the first subperiod explains a larger proportion of the total security variance than it explains during the second subperiod. The results of Table 11.6 also reveal a negative relationship between the proportion of the total security variance accounted for by the first factors and the group size. In addition there exists a negative relationship between the proportion of the total security variance accounted for by each of the remaining common factors and the group size. However, the amount of the total security variance accounted for by the first factor decreases with the group size more than the amount of the total security variance accounted for by each of the remaining common factors.

This justifies the positive relationship between the number of factors and the group size and it shows the similarity of the results with those reported in Tables

Table 11.6 An example indicating the importance of the first factor across two nonoverlapping subperiods .
Sample B : Period :11/1956-12/1981 .

SUBPERIOD	GROUP SIZE [1]	NUMBER OF COMMON FACTORS	PROPORTION OF TOTAL VARIANCE ACCOUNTED FOR BY THE FIRST COMMON FACTOR	PROPORTION OF TOTAL VARIANCE ACCOUNTED FOR BY THE REMAINING COMMON FACTORS	COMMULATIVE PROPORTION OF TOTAL VARIANCE ACCOUNTED FOR BY THE REMAINING COMMON FACTORS
11/1956-5/1969	5	1	51.9	0	0
	10	1	46.8	0	0
	15	2	42.9	8.8	8.8
	20	3	39.3	6.9,6.0	12.9
	40	9	37.3	4.5,4.2,3.7,3.1, 2.9,2.7,2.6,2.5	28.9
6/1969-12/1981	5	1	56.4	0	0
	10	2	54.6	10.9	10.9
	15	3	52.8	7.7,6.3	14.0
	20	4	50.6	5.9,5.0,4.3	15.2
	40	10	48.8	4.2,3.9,3.1,3.0, 2.7,2.4,2.2,2.2, 2.0	25.6

1 The master group used in this example is the first out of the 5 master groups of sample B .

10.5, 10.6, 11.5 and 11.6.

Next comparing the results of samples A and B a number of points are worth noting :

(i) The invalidity of the assumption that the number of factors remains unchanged across various time periods for the same group of securities and for different security groups was obtained by using different subperiods comprised of 60, 100 and 151 monthly observations. Hence the consideration of the subperiods of both samples implies that the produced results about the present assumptions are more powerful and reliable.

(ii) By utilizing sample A, the subperiods 1/1972 - 12/1976 and 1/1977 - 12/1981 and groups of 42 securities it was found that the existence of a multi-factor linear model is rejected for each value of K (= number of factors).

This may be attributed to the multicollinearity between the returns on securities, because it contains securities with observable returns over a long time period. Indeed Tables 11.2 and 11.3 show that for security groups of size 40 there exist multi-factor linear models describing the security returns.

Summarizing the conclusions from Tables 11.1, 11.2 and 11.3 it can be inferred that :

(1) The number of factors which influence the security returns changes across various time periods for the same group of securities and for different security

groups.

(2) The number of factors affecting the security returns changes across various groups of different sizes and across various groups of the same size for the same subperiod.

11.2 Some Possible Explanations of the Results

The findings derived in Section 11.1 by using sample A produce the following procedural bias :

There are cases where the security returns generating model is rejected for any positive degrees of freedom.

A possible explanation of this bias could be the deviation from normality in the joint distribution of security returns. However, the findings of Chapter 9 reveal that this is not the case, because the joint distribution of security monthly returns is close to normal. Moreover, if the joint distribution of security monthly returns deviates from a normal distribution Jöreskog (1963) provided evidence inticating that the chi-square test for the goodness of fit of the factor model is fairly robust against such moderate departures.

Another potential explanation of this bias can be attributed to Rao's factor analytic procedure employed in this study. In the following pages this argument will be explained.

(I) Rao's Factor Analysis and the Extreme Values of the Correlation Matrix's Determinant

The determinant of the correlation matrix is always in the range one to zero. It attains the maximum value of one if all the non-diagonal elements of the correlation matrix are zero ; while it attains the value of zero if all the non-diagonal elements of the correlation matrix are equal to plus unity (perfect positive correlation).

The aim of factor analysis is to explain the correlations between a large set of variables by introducing a minimal number of underlying factors that account for the correlations. So, if all the non-diagonal elements of the correlation matrix are zero (or close to zero) then the data are not suitable for factor analysis (see also the ninth test for the complete independence of the correlation matrix of security returns). However, in this work it was found that the non-diagonal entries of the correlation matrices are significantly different from zero. Thus the determinants of the correlation matrices are different from one.

On the other hand, if all the non-diagonal elements of the correlation matrix are equal to plus unity, then only one factor can be extracted from the correlation matrix. This can be shown as follows :
Suppose the following (N x 1) correlation matrix is considered :

$$R_1 = \begin{bmatrix} 1 & 1 \dots & 1 \\ 1 & 1 \dots & 1 \\ \dots & \dots & \dots \\ 1 & 1 \dots & 1 \end{bmatrix}$$

This matrix is singular with rank equal to one.

Furthermore the characteristic equation of R_1 can be expressed as :

$$\det (R_1 - \lambda_i I) = 0$$

where

$i = 1,2,\dots,N.$

λ_i = the i^{th} characteristic root of R_1 .

$I = (N \times N)$ unit matrix.

The last equation can be written equivalently as follows :

$$\lambda^{N-1}(\lambda - N) = 0 \qquad (11.1)$$

Equation (11.1) has N-1 characteristic roots equal to 0 and only a single eigenvalue different from zero. Hence there exists only a single factor extracted from the correlation matrix R_1.

Furthermore one can write

$$R_1 = B_1 B_1' \qquad (11.2)$$

where

B_1 = the (N x K) matrix of the factor loadings and B_1' is the transpose of B_1.

Equation 11.2 implies

$$\text{rank } (R_1) = \text{rank } (B_1 B_1')$$

But the rank of the product of the two mutual transpose matrices B_1 and B_1' is equal to the rank of B_1. Hence it can be concluded that the rank of the matrix B_1 is equal to 1. Therefore, in view of equation 11.2 the factor loadings of the variables on the single factor are equal to one.

Finally, the proportion of total variance accounted by the single factor is

$$\frac{\sum_{i=1}^{N} b_{i1}^2}{N} = 1$$

where

b_{i1} = the factor loading of the variable i on the single factor.

However, in this study all the variables are less than perfectly positively correlated and hence the correlation matrices have determinants different from zero.

If $R = (p_{ij})$, where $i,j = 1,2,\ldots,N$, is the correlation matrix of security returns, then in view of the previous mentioned analysis it can be inferred that the importance of the first extracted factor increases as p_{ij}, where $i \neq j$, increases. This argument may be a possible explanation of the results of Tables 11.4, 11.5 and 11.6, where the proportion of total security variance accounted by the first factor in the subperiods 1/1972 - 12/1976, 7/1973 - 10/1981 and 6/1969 - 12/1981 respectively, is larger than the proportion of total security variance accounted for by the first factor in other subperiods.

(II) The Sample Size in Terms of Time Periods and the Correlation Matrix's Determinant

Since the correlation matrix's determinant attains the maximum value when the non-diagonal elements of the correlation matrix are zero, it can be inferred that the determinant magnitude decreases as the non-diagonal entries of the correlation matrix increase. Viewed in another way the magnitude of the correlation matrix's determinant is dependent on the magnitudes of the correlations among the security returns. If the security returns are highly correlated (multicollinear) then the value of the determinant is very small and it is greater than zero. Moreover, if the security returns are not highly correlated the value of the determinant is not small and it is less than one.

On the other hand, the magnitude of the determinant of the correlation matrix, in general, may be dependent on the following two factors :

(1) The dimension of the correlation matrix.

(2) The length of the sample period over which the entries of the correlation matrix are estimated.

In this study it was found that the correlation determinant's magnitude was an increasing function of the group size. Table 11.7 presents an example of such a situation by considering the first master group of sample A and its (overlapping) subgroups.

It was also found that, in general, the magnitude of the correlation matrix's determinant increases with the length of the sample period. There were, however, few cases where the determinant of the correlation matrix was decreased as the sample period's length was increased. But it was always observed for the sample period of the smallest length (i.e. 60 security monthly return observations), that the determinants of the correlation matrices were very small. This was due to the high correlations between the security returns. For example, there were cases where the correlations among security returns were in the range of .75 to .86 .

(III) The Sample Size in Terms of Time Periods and Rao's Test for Goodness of Fit of the Factor Model

It is noted that the correlation matrix's determinant is equal to the product of its eigenvalues. Therefore, a small determinant indicates that some of the eigenvalues of the correlation matrix are very small.

Table 11.7 An example indicating the relationship between the group size and the magnitude of the correlation matrix's determinant.
Sample A : Period : 1/1972-12/1981.

NUMBER OF OBSERVATIONS PER SECURITY : 120	
GROUP SIZE	DETERMINANT OF THE CORRELATION MATRIX
5	.2817
10	.4451 -01
15	.2925 -02
21	.8761 -04
26	.3542 -05
31	.6387 -07
36	.1246 -08
42	.3070 -11

An equivalent expression of equation (8.24) is given by

$$C_3=\left(T_1-1-\frac{2N+5}{6}-\frac{2}{3}K\right)\ln\frac{\lambda_{K+1}\cdots\lambda_N}{\left[\dfrac{\lambda_{K+1}+\cdots+\lambda_N}{N-K}\right]^{N-K}} \qquad (11.3)$$

Since the chi-square distribution is always positive the quantity $\ln\dfrac{\lambda_{K+1}\cdots\lambda_N}{\left[\dfrac{\lambda_{K+1}+\cdots+\lambda_N}{N-K}\right]^{N-K}}$ has to be positive.

This quantity is positive if $\lambda_{K+1}\cdots\lambda_N > \left[\dfrac{\lambda_{K+1}+\cdots+\lambda_N}{N-K}\right]^{N-K}$.

Furthermore when the sample period's length is small and the group size is large (e.g. 60 monthly observations per security and a group size of 42) the determinant of the correlation matrix is very small. For example, the determinant of the first master group of sample A, when 60 security return monthly observations are taken into account, is .1041 - 21.

The determinant of the symmetric matrix GRG is equal to the product of the determinants $|G|$, $|R|$ and $|G|$ (G = a diagonal matrix whose diagonal elements are given by equation (8.22) and R is the sample correlation matrix of security returns).

Therefore the determinant of the matrix GRG is also very small. But a known property of determinants gives :

$$\lambda_{K+1} \cdots \lambda_N = \frac{|GRG|}{\lambda_1 \cdot \lambda_2 \cdots \lambda_K}$$

where

$|GRG|$ = the determinant of the matrix GRG.

$\lambda_1, \ldots, \lambda_K$ = the K largest characteristic roots of the matrix GRG.

Consequently, a small $|GRG|$ produces a very small product $\lambda_{K+1} \cdots \lambda_N$.

Moreover the quantity $\left[\frac{\lambda_{K+1} + \ldots + \lambda_N}{N - K}\right]^{N-K}$ is very small which in turn implies a very large value of the fraction

$$\frac{\lambda_{K+1} \cdots \lambda_N}{\left[\frac{\lambda_{K+1} + \ldots + \lambda_N}{N - K}\right]^{N-K}}$$ and hence a high value of

C_3(see equation (8.24)) . That is a high value of C_3 is implied from a low correlation matrix's determinant and a low correlation matrix's determinant is implied from a small number of observations per security and a large group size . Thus a high value of C_3 is implied when the difference between the sample size per security and the group size is very small. In such a case the value of C_3 cannot approximate a chi-square distribution and hence no admissible value of K (=number of factors) gives a satisfactory fit. An example of this situation is shown in Table 12.5. For this example the first master group of sample A is utilized. Such a group cannot produce more than 33 factors since the chi-square statistic requires positive degrees of freedom.

Thus, in view of the foregoing discussion, it can be inferred that the bias presented in this chapter may be attributed to the inability of Rao's factor analytic procedure to cope with small samples in terms of time periods. This implies large values of C_3, and thus in order to produce a value of C_3 which approximates a chi-square's distribution value a larger number of factors is required.

By taking into consideration the analysis of this section it can be concluded that the chi-square distribution of C_3 can be probably trusted if :

$$T_1 \geqslant N + 40$$

where

T_1 = the sample size in terms of time periods.

N = the number of securities per group.

By using sample A and B it was found that the number of factors changes across the various time periods. This may be due to disadvantages of the sequential procedure ; in view of such a procedure the critical value of the test criterion is fixed, while the null hypothesis regarding the number of factors is being

Table 11.8 An example indicating the rejection of a K-factor model for all values K(= number of factors).
Sample A: Period 1/1972-12/1981 .

NUMBER OF OBSERVATIONS PER SECURITY : 60			
NUMBER OF FACTORS	x^2	DEGREES OF FREEDOM	CRITICAL VALUES FOR x^2 DISTRIBUTION AT THE 99% LEVEL OF CONFIDENCE
6	1044.0	624	709.1
7	950.1	588	670.7
8	894.5	553	633.3
9	834.8	519	596.8
10	782.7	486	561.4
11	763.3	454	527.0
12	699.5	423	493.5
13	667.2	393	461.1
14	623.0	364	429.7
15	593.3	336	399.2
16	554.4	309	369.7
17	521.0	283	341.2
18	485.3	258	313.7
19	458.7	234	287.2
20	401.3	211	261.1
21	382.4	181	237.1
22	359.1	168	213.5
23	318.3	148	190.9
24	281.6	129	170.4
25	260.4	111	149.7
26	240.0	94	128.8
27	212.2	78	109.9
28	186.7	63	92.0
29	172.0	49	74.9
30	147.7	36	58.6
31	127.6	24	42.9
32	108.1	13	27.6
33	88.0	3	11.3

tested in sequence. As a result different chi-square values are produced and thus factors representing only statistical artifacts are emerged. In such a case the derived results are unrealistic.

Except for the possible explanations regarding the mathematical procedure followed for testing the stability of the number of factors across various time periods, there are also some other possible explanations of the results. In the real world it is possible that some of the factors found to affect the security returns in one period to be unimportant in the following period. In this case the number of factors determining the security returns changes through time. A political crisis, an oil crisis, a war scare,etc., are some examples of factors found to be critical in one period but unimportant in the following period.

The reported results in this chapter indicate that for the subperiods which include the 1974 - 1975 period, security groups yield a larger number of factors relative to the number of factors yielding in other subperiods. This may be attributed to the existence of some important factors affecting the security returns during the period 1974 - 1975 (e.g. oil crisis), but having no influence on the security returns during other periods.

Furthermore, it is likely to be found in the real world that there are some factors determining the security returns over short time periods, while they are unimportant over long time periods. This is a possible explanation of some of the results presented in this chapter, where the relevant number of factors decreases as the number of observations per security increase.

Finally, the consideration of a large sample in terms of time periods probably implies that the number of factors determining the security returns for the same security groups are influenced by different economic cycles. As a consequence different factors will affect the security returns across various time periods.

11.3 Comparison with Previous Studies

The stability of the number of factors across different time periods for the same group of securities has been empirically investigated by Kryzanowski and Chau (1982). Their results indicated that, on average, the number of factors does not change substantially across various samples in terms of different period lengths for the same group of securities. Moreover the larger the sample size in terms of different period lengths, the smaller the number of factors which affect the security returns.
Kryzanowski and Chau attributed their latter results to Rao's factor analytic method that they used. But they did not give a clear explanation of the reasons that produced their results. Their latter results are very similar to the results reported in this study.

However, they utilized in some of treir tests groups containing 60 securities and sample sizes per security equal to 60 observations. In this case they found a large number of factors compared with the number of factors emerged from cases where they used a larger number of observations. The discussion mentioned in the previous section can be considered as a possible justification of their results.

Furthermore, Kryzanowski and Chau reported cases where the average number of relevant factors was equal to 20. These findings, however, does not imply that every group in their sample produced 20 factors. There could be groups yielding a number of factors greater than 20. If this was the case it may also be justified by the small number of observations used relative to the group size.

Finally, Hughes (1982) used two groups containing 110 securities and a sample size of 120 observations. Her tests can be criticized because she utilized a large group size relative to the number of observations per security. Hughes stated :

> "The number of factors extracted was increased from five to twelve and the chi-square statistic continued to indicate that many additional factors were needed for adequate factoring ". (p.16).

But,the number of factors increases with the group size and the chi-square test she used requires a large number of observations relative to the size of group. Therefore in her case the K-factor generating model could probably be rejected for every possible value of K (= the number of factors).

11.4 The Implications of the Empirical Results

In the light of the results presented in this chapter it can be deduced that Rao's factor analytic technique yields for the London Stock Exchange different security returns generating models across various time periods for the same group of securities. It was pointed out in Section 11.2 that such results may be due either to Rao's factor analytic method or to the existance of some factors affecting the security returns in one period but being unimportant in the following period.

The results derived in this chapter indicate that the security returns generated model cannot be used for forecasting purposes.

Since it was found different returns generating models across various time periods for different security groups, it can be asserted the violation of the A.P.M.'s assumption about the uniqueness of the security returns generating model across various time periods for the same group of securities or for different security groups. In view of the identification problem of the security returns generating model of the A.P.M., however, there is no way to ascertain which is the appropriate time length that has to be used in order to examine empirically the validity of the A.P.M. By utilizing a given sample period it cannot be asserted that the producing security returns generating model is the unique model of the A.P.M., since if such a model exists it cannot be identified.

Furthermore, the instability of the number of factors through time shows the violation of a major assumption required to transform the A.P.M. into a testible relationship. Therefore it can be inferred that the A.P.M. cannot be tested unambiguously using time series data for the London Stock Exchange. As a consequence the introduction of the A.P.M. into the literature as a testible alternative to the C.A.P.M. may be challenged.

On the other hand, the utilization of a small sample in terms of time periods implies that there is not a linear model describing the returns on securities. This shows a fundamental weakness of the test for the goodness of fit of the factor model and reveals the inability of Rao's factor analysis solutions to describe security returns generating models.

Finally, it may be stated that a test concerning the intertemporal stationarity of the factor beta coefficients will be useless, since the factors affecting the security returns are not the same across various time periods for the same group of securities.

11.5 Conclusions

The number of factors does not remain unchanged across various time periods for the same group of securities and for various security groups of different sizes. Consequently the security returns generating model can not be used for forecasting purposes.

Moreover the A.P.M. cannot be tested unambiguously using time series data from the London Stock Exchange.

Finally, in order to rely on Rao's factor analytic results the difference between the number of observations and the group size must be at least equal to 40.

CHAPTER 12

SUMMARIZED CONCLUSIONS OF THE RESULTS AND SUGGESTIONS FOR FURTHER RESEARCH

This chapter gives a summary of the results and conclusions derived from an empirical examination of the assumptions required to produce an unambiguous test of the A.P.M. for the London Stock Exchange. It is also concerned with some recommendations for future research.

12.1 Summary and Conclusions

The present study has concentrated upon an empirical verification of the assumptions which ensure an unambiguous test of the A.P.M. using time series data from the London Stock Exchange.

The first assumption maintains that, the distributions of security monthly returns are normal, as well as the security mean returns and the covariance or correlation matrix of security returns are stationary through time The results of this research indicate that the distributions of security monthly returns are approximately normal during the various subperiods studied. Moreover it was found that the security mean returns are stationary through time.

Lastly, the reported findings failed to support the intertemporal stationarity of the covariance matrix of security returns, but they supported the intertemporal stationarity of the correlation matrix of security returns.

As a consequence the distributions of security returns are not intertemporally stationary and thus for the tests of the A.P.M. the correlation matrix of security returns should be factor analyzed.

Also it is noted that previous U.K. studies relied upon the assumption concerning the intertemporal stationarity of the joint distribution of security returns should be interpeted with caution .

The second and third assumptions maintain that, the number of factors affecting the security returns remains unchanged across security groups of different sizes and across various security groups of the same size.

The evidence presented in this study reveals that the utilization of Rao's factor analytic technique produces different number of factors across various groups of different sizes and across various groups of the same size.

Hence, in view of the identification problem of the unique security returns generating model of the A.P.M., tests of the A.P.M. utilizing U.K. data are not necessarily tests of the model. The A.P.M. may be held, but the employment of the existing methodology does not ensure an unambiguous test of the model for the London Stock Exchange.

The fourth and fifth assumptions maintain that, the number of factors affecting the security returns remains unchanged across various time periods for the same group of securities and across various time periods for security groups of different sizes.

The reported results show that Rao's factor analytic method yields different number of factors across various time periods for the same group of securities and across various time periods for security groups having different sizes. Therefore the security returns generating model cannot be utilized for predicting purposes.

In addition according to the identification problem of the unique security returns generating model of the A.P.M., tests of the A.P.M. utilizing a given number of monthly security returns observations from the London Stock Exchange are not necessarily tests of the model. <u>The A.P.M. may be valid, but the utilization of the existing methodology does not ensure an unambiguous test of the model using time series data from the London Stock Exchange.</u>

12.2 Recommendations for Future Research

The empirical results and the conclusions presented in this study have pointed to a number of possibilities where further research can be carried out. The aim of this section is to summarize these research possibilities.

1. In the present study the security master groups were formed randomly; the securities were listed in ascending order and then they were ordered into master

groups of 42 securities. However,it was pointed out in this study that the positive relationship between the number of common factors and the group size may be arise because of the way in which the samples of securities were ordered. (see Section 10.4). Therefore future research should attempt to test the assumption concerning the relationship between the number of factors and the group size by ordering securities according to their industry classification.

2. The A.P.M. does not specify whether the security returns have to be nominal or real . Since real returns help to eliminate nonstationarities due to changes in the rate of inflation, further research should be carried out in order to verify whether the A.P.M. can be tested unambiguously by using real rates of returns.

3. Development and testing of a model which overcomes the criticisms of the C.A.P.M.,especially the market portfolio's identification problem, and contains observable factors determining the security returns.
A multi-factor linear regression model with beta coefficients moving randomly through time may be a satisfactory model explaining the variation in security returns. Such a model is called a random coefficient multi-factor model and it has been first developed by Theil and Mennes(1959) and subsequently axtended by Swamy(1970).

Hence by taking into considaration the following:

(i) The problems emerging from the utilization of an unobservable security returns generating model,and

(ii) A security returns generating model would be considered as a satisfactory model if there exist few common factors determining the security returns and these factors' variability explains a large proportion of the total variabilities on security returns.

it can be suggested a model comprised of five common factor.

The common factors might be:

(1) A market index.

(2) A industry index.

(3) A bond index.

(4) The gross national product.

(5) The inflation rate.

Assuming that the security returns are affected by these common factors and using the methodology of Theil and Mennes and Swamy it can empirically examined the following:

(I) Whether the common factors provide enough information to explain the intercorrelations among the security returns.

(I) Whether this multi-factor security returns generating model does a better job than a single-index security returns generating model.

(III)Which are the equilibrium implications of this multi-factor security returns generating model.

(IV) If such a security returns generating model implies an asset pricing model then empirically one should verify the assumptions which ensure an unambiguous test of the model using a time series data.

Finally, it can be noted that in testing a model of the type described previously there may exist some problems, e.g. the existence of some multicollinear factors. However, it is worthwhile to try out the above idea which seems more promising than testing the realism and potencial usefulness of a model relied upon an unobservable number of factors.

APPENDIX A

The Proof of the Arbitrage Pricing Model

This appendix presents the mathematical proof of the A.P.M. under Ross' "no arbitrage" condition and under the weaker "no arbitrage" condition. The appendix begins with some results from the linear algebra field.

Let (V_1,f) be an N-dimensional real inner product space over the set of real numbers $\mathbb{R}$, where

$$f : V_1 x V_1 \longrightarrow \mathbb{R} : (z_1, z_2) \longrightarrow f(z_1, z_2) = z_1' z_2 \;.^1$$

N =a finite integer.

Let C and S_1 be two proper and non-empty subsets of V_1 such that;

$$C = \left\{ i,\ {}^1b,\ {}^2b, \ldots,\ {}^Kb \right\}$$

$$S_1 = \Big\{ z_{m1} \in V_1 : z_{m1} = \lambda_{m1} i + B\Lambda_{m1}, \lambda_{m1} \in \mathbb{R}, \Lambda_{m1} \in \mathbb{R}^K, 1 \le m \le \zeta, \zeta \in \mathbb{N},$$
$$|\lambda_{kl}| \neq 0, k=1,2\ldots,K+1,\ K+1 < \zeta, i, {}^1b, {}^2b, \ldots, {}^Kb \in C \Big\}^2$$

The conclusion that follows gives a very important property of S_1.

---

1 In this appendix transposition of vectors or matrices are denoted by " ' " .

2 i is the (Nx1) unit vector, $\mathbb{N}$ is the set of finite integers and $|\lambda_{kl}|$ is the determinant of the (K+1)x(K+1) matrix $[\lambda_{kl}]$.

<u>Conclusion A.1</u> S_1 is a (K+1)-dimensional subspace of V_1.

<u>Proof</u>

S_1 is the set of all linear combinations of $i, {}^1b, {}^2b, \dots, {}^Kb$, where $i, {}^1b, {}^2b, \dots, {}^Kb \in V_1$. Hence S_1 is a subspace of V_1.[1]

Next it is proved that $B_s = \{z_{11}, z_{21}, \dots z_{K+11}\}$ is a basis of S_1, where $z_{k1} \in S_1$, $k=1,2,\dots,K+1$.

Suppose

$$d_{11}z_{11} + d_{21}z_{21} + \dots + d_{K+11}z_{K+11} = 0 \qquad (A.1)$$

where $d_{11}, d_{21}, \dots, d_{K+11} \in \mathbb{R}$.

Equation (A.1) can be rewritten as

$$d_{11}(\lambda_{11}i + B\Lambda_{11}) + d_{21}(\lambda_{21}i + B\Lambda_{21}) + \dots + d_{K+11}(\lambda_{K+11}i + B\Lambda_{K+11}) = 0$$

It is easily establish, by manipulating the last equation, that

$$(d_{11}\lambda_{11} + d_{21}\lambda_{21} + \dots + d_{K+11}\lambda_{K+11})i + (d_{11}\Lambda_{11} + d_{21}\Lambda_{21} + \dots + d_{K+11}\Lambda_{K+11})B = 0$$

But the rank of the matrix $\begin{bmatrix} B & i \end{bmatrix}$ is K+1 (see assumptions of the A.P.M.). Therefore it can be concluded that

$$d_{11}\lambda_{11} + d_{21}\lambda_{21} + \dots + d_{K+11}\lambda_{K+11} = 0$$

$$d_{11}\Lambda_{11} + d_{21}\Lambda_{21} + \dots + d_{K+11}\Lambda_{K+11} = 0$$

or, equivalently

$$D'\left[\lambda_{k1}\right] = 0 \qquad (A.2)$$

---

1 For the algebraic treorems used in this appendix see Nering(1970, Chapter I).

where

D = a 1x(K+1) column vector with elements the real numbers $d_{11}, d_{21}, \ldots, d_{K+11}$.

The (K+1)x(K+1) matrix $[\lambda_{kl}]$ is nonsingular. Hence the linear homogeneous system represented by equation A.2 has the trivial solution $D = \underline{0}$,where $\underline{0}$ is the 1x(k+1) zero vector. Therefore the vectors $z_{11}, z_{21}, \ldots z_{K+11}$ are linearly independent. Next it is shown that each vector in S_1,which is not a member of B_s,can be obtained by a linear combination of the vectors in B_s. That is each $z_{m1} \in S_1$,where $z_{m1} \notin B_s$,has the form

$$z_{m1} = g_{11} z_{11} + g_{21} z_{21} + \ldots + g_{K+1} z_{K+1} \tag{A.3}$$

with $g_{11}, g_{12}, \ldots, g_{K+11} \in \mathbb{R}$.

Notice that

$$g_{11} z_{11} + g_{21} z_{21} + \ldots + g_{K+11} z_{K+11}$$
$$= g_{11}(\lambda_{11} i + B \Lambda_{11}) + g_{21}(\lambda_{21} + B \Lambda_{21}) + \ldots + g_{K+11}(\lambda_{K+11} + B \Lambda_{K+11})$$
$$= (g_{11}\lambda_{11} + g_{21}\lambda_{21} + \ldots + g_{K+11}\lambda_{K+11}) i$$
$$+ (g_{11}\Lambda_{11} + g_{21}\Lambda_{21} + \ldots + g_{K+11}\Lambda_{K+11}) B$$

Since $B_s \not\ni z_{m1} = \lambda_{m1} + B\Lambda_{m1}$ one can set

$$g_{11}\lambda_{11} + g_{21}\lambda_{21} + \ldots + g_{K+11}\lambda_{K+11} = \lambda_{m1}$$
$$g_{11}\Lambda_{11} + g_{21}\Lambda_{21} + \ldots + g_{K+11}\Lambda_{K+11} = \Lambda_{m1}$$

The last K+1 equations can be considered as a linear system of equations having K+1 unknowns. Since $|\lambda_{kl}| \neq 0, k=1,2,\ldots,K+1$, such a system has a unique non-zero solution. That is z_{m1} can be expressed as a lineal combination of the vectors in B_s.

Thus S_1 is generated(or spanned) by B_s. As a consequence B_s is a basis of S_1. Hence S_1 is a (K+1)-dimensional subspace of V_1.

Q.E.D.

The orthogonal complement of S_1 in V_1 is defined by

$$S_1^{\perp} = \{ x \in V_1 : x' z_{m1} = 0 \text{ for all } z_{m1} \in S_1 \}$$

The dimension of S_1 is N-(K+1) . Furthermore the orthogonal complement of S_1 in V_1 is defined by

$$S_1^{\perp\perp} = \{ {}^1h \in V_1 : {}^1h' x = 0 \text{ for all } x \in S_1 \}$$

Since the vector space V_1 has a finite dimension it can be inferred

$$S_1 \oplus S_1^{\perp} = V_1 \tag{A.4}$$

$$S_1^{\perp\perp} = S_1 \tag{A.5}$$

The next theorem proves the A.P.M.

<u>THEOREM A.1</u> Under the assumptions of the K-Factor A.P.M. the expected return vector, R_E, can be approximately expressed as a linear combination of i and B. That is

$$R_E \approx \lambda_{11} i + B\Lambda$$

where λ_{11} and the members of the (Kx1) column vector Λ are non-zero real numbers.

<u>Proof</u>

Assume that al is a well diversified arbirtage portfolio .

Namely

$$y'_{a1}i = 0 \qquad (A.6)$$

Without loss of generality it can be assumed that y_{a1} can be expressed as in equation (4.8). If equation (4.1) be premultiplied by y'_{a1} the addiotional return that can be gained from the arbitrage portfolio al may be written by means of equation (A.7) below

$$y'_{a1}\tilde{R} = y'_{a1}R_E + (y'_{a1}B)\tilde{\delta} + y_{a1}\tilde{e} \qquad (A.7)$$

The portfolio al is a well diversified portfolio. So in view of Definition 4.3 (see page 75) N is large and the members of the vector e are mutually independent. Moreover by means of the A.P.M.'s assumption $(1)_{iii}$ the elements of the vector $\tilde{e}$ are commonly distributed with expectations equal to zero. Then one may apply the law of large numbers[1] to conclude

$$y'_{a1}\tilde{e} = 0 \;^{2} \qquad (A.8)$$

The variance of the term $y_{a1}e$ is equal to zero. This can be proved as follows:

$$\mathrm{Var}(y'_{a1}e) = \frac{1}{N^2}\,\mathrm{Var}\left(\sum_{i=1}^{N}\tilde{e}_{it}\right)$$

$$= \frac{1}{N^2}\sum_{i=1}^{N}\sigma^2_{e_{it}}$$ (Since the security disturbances are mutually independent)

1 The law of large numbers can be stated as follows: If S_i is a sequence of mutually independent variates with a common disrtibution and if the expectation $\mu = E(S_i)$ exists, then for every $\varepsilon > 0$ as $N \to \infty$ the probability

$$\Pr\left\{\left|\frac{S_1+S_2+\ldots+S_I}{N} - \mu\right| > \varepsilon\right\} \to 0$$

(Kendall and Buckland (1978)).

2 Equation (A.8) indicates that $y'_{a1}\tilde{e}$ approaches to zero,but it is not exactly equal to zero. If $y'_{a1}\tilde{e}$ was equal to zero then it should exist interdependencies among the disturbances of different securities. This is not consistent with the A.P.M.'s assumption (1)iii.

where

$\sigma^2_{e_{it}}$ = the variance of e_{it}.

Let $\sigma^2 = \bar{\sigma}^2_{e_{it}} = \frac{\sum_{i=1}^{N} \sigma_{e_{it}}}{N}$. Then

$$\mathrm{Var}(y'_{a1}\tilde{e}) = \frac{\sigma^2}{N}$$

Hence as N grows large

$$\mathrm{Var}(y'_{a1}\tilde{e}) \approx 0$$

Viewed in another way, as N grows large the idiosyncratic risk of the portfolio al can be eliminated.

Next suppose that y'_{a1} is selected so that the following condition is met

$$y'_{a1}B = 0 \tag{A.9}$$

By virtue of equations (A.8) and (A.9) equation (A.7) can be written in the form

$$y'_{a1}R \approx y'_{a1}R_E \tag{A.10}$$

The portfolio al uses no wealth and it was selected so that to have no risk (i.e. it has neither systematic nor idiosyncratic risk). Accordingly by the A.P.M.'s assumption (8) it can be deduced that the portfolio al has to have an expected return approaching to zero.

That is

$$y'_{a1}R_E \approx 0 \tag{A.11}$$

Therefore it has been proved that the equality

$$y'_{a1}i = y'_{a1}B = 0 \quad \text{implies} \quad y'_{a1}R_E \approx 0 \quad .$$

Hence $y_{a1} \in S_1$ and $y'_{a1}R_E$ 0. As a result the definition of $S_1^{\perp\perp}$ can be used to infer that R_E is a member of $S_1^{\perp\perp}$. Bearing in mind equation (A.5) one concludes that R_E is also a member of S_1. Consequently R_E is generated by $C = \{i, {}^1b, {}^2b, \ldots, {}^Kb\}$. Thus the following approximation is asserted

$$R_E \approx \lambda_{11} i + B\Lambda^1 \tag{A.12}$$

Q.E.D.

The proof of the A.P.M. is suffered from the following problem: According to the linear algebra R_E is a member of $S_1^{\perp\perp}$ if $y'_{a1}R_E$ is exactly equal to zero. However, in the previous mentioned proof $y'_{a1}R_E$ approaches to zero.

An immediately result of Theorem A.1 is the following:

<u>COROLLARY A.1</u> Suppose that $L_1, L_2, \ldots, L_K$ are K portfolios defined by the (Nx1) column vectors $X_{L1}, X_{L2}, \ldots, X_{LK}$, respectively, with

$$X'i = i_1 \tag{A.12}$$

$$X'B = I \tag{A.13}$$

where,

i_1=the (Kx1) unit vector.

X = the (NxK) matrix whose columns are $X_{L1}, X_{L2}, \ldots, X_{LK}$.

1 Roll(1977) proved Theorem A.1. In this appendix the proof of Ross was presented in a more detailed fashion. In this proof the well diversified arbitrage portfolios played an important role, because in such a case the law of large numbers is applicable. Huberman, however, demostrated that it is not necessary to use a well diversified portfolio as a tool to prove the A.P.M. He argued that one can use a portfolio, which is defined by a column vector that satisfies two orthogonality conditions (pp.6-7).

Then the expected return vector can be descrided as follows:

$$R_E \approx r_Z i + B(^1R - r_Z i_1)$$

where

r_Z=the expected return on a portfolio,with

$$X_Z' B = 0 \; ^1 \qquad (A.14)$$

^1R=the (Kx1) column vector with elements the expected returns on portfolios L1,L2,...,LK .

Proof

Premultiplying equation (A.12) by X_Z' it can be seen that

$$X_Z' R_E \approx \lambda_{11} X_Z' i + X_Z' B \Lambda \qquad (A.15)$$

Since $X_Z' B=0$ and $X_Z' i=1$,equation (A.15) can be expressed as

$$r_Z = \lambda_{11} \qquad (A.16)$$

Substituting λ_{11} from equation(A.16) into equation (A.12) it can be established that

$$R_E \approx r_Z i + B\Lambda \qquad (A.17)$$

Premultiplying both sides of equation (A.17) by X' it follows immediately that

$$X' R_E \approx r_Z X' i + X' B \qquad (A.18)$$

However,since $X' i = i_1$ and $X' B = I$ equation (A.18) can be rewritten as

$$^1R \approx r_Z i_1 + \Lambda \qquad (A.19)$$

1 If there exists a riskless security, then r_Z is the riskless rate of interest.

Solving for Λ in equation (A.19) gives

$$\Lambda \approx {}^{1}R - r_Z i_1 \quad {}^{1} \tag{A.20}$$

Finally by means of equation (A.20) it is evident that equation (A.17) is expressible as

$$R_E \approx r_Z i + B({}^{1}R - r_Z i_1) \tag{A.21}$$

The next corollary follows directly from corollary A.1 .

<u>COROLLARY A.2</u> Suppose pl is a portfolio whose return is given by equation (4.3) . Then

$$r_{pl} \approx r_Z + b_{pl}({}^{1}R - r_Z i_1) \tag{A.22}$$

where

$$b_{pl} = X_{pl} B \tag{A.23}$$

<u>Proof</u>

Multiplying equation (A.21) by X'_{pl} and using equation (A.23) one can easily produce equation (A.22).

Q.E.D.

Next it is proved the A.P.M. under the weaker "no arbitrage" condition.

<u>THEOREM A.2</u> Suppose that the assumption of Ross' "no arbitrage" condition is substituted by the weaker "no arbitrage" condition. Then R_E can be approximately expressed as follows:

---

1 The risk aversion assumption of the A.P.M. implies that investors require greater returns from more risky portfolios. Concequently the elements of the vector Λ are positive.

$$R_E \approx \lambda_{11} i + B\Lambda$$

where λ_{11} and the elements of the (Kx1) column vector Λ are non-zero real numbers.

Proof

Suppose that X_{p1} and X_{p2} are the investment proportions vectors defining two well diversified portfolios p1 and p2, respectively, where

$$X'_{p1} i = 1 \tag{A.24}$$

$$X'_{p2} i = 1 \tag{A.25}$$

Equations (A.24) and (A.25) easily produce

$$(X'_{p1} - X'_{p2}) i = 0$$

The last equation can be written in the equivalent form

$$X'_p i = 0 \tag{A.26}$$

where

$$X_p = X_{p1} - X_{p2} \tag{A.27}$$

Without loss of genelality one can choose X_{p1} and X_{p2} such that

$$X_p = \begin{bmatrix} \frac{1}{N} \\ \vdots \\ \frac{1}{N} \\ -\frac{1}{N} \\ \vdots \\ -\frac{1}{N} \end{bmatrix} \tag{A.28}$$

where the number of the positive weigths is taken equal to the number of the negative weights.

Premultiplying both sides of equation (4.1) by X_p' yields

$$X_p'R = X_p'R_E + (X_p'B)\delta + X_p'e \tag{A.29}$$

An application of the law of large numbers shows that

$$X_p'e \approx 0$$

and hence

$$Var(X_pe) \approx 0$$

So equation (A.29) can be written in the following manner

$$X_p'R \approx (X_{p1}'R_E - X_{p2}'R_E) + (X_{p1}'B - X_{p2}'B) \tag{A.30}$$

Furthermore choose X_{p1} and X_{p2} such that

$$X_{p1}'B = X_{p2}'B = \xi \in \mathbb{R}_o^+ \text{ [1]}$$

which in turn is equivalent to

$$X_p'B = 0 \tag{A.31}$$

Accordingly making use of the weaker "no arbitrage" condition it may be concluded

$$X_p'R_E \approx 0 \tag{A.32}$$

By taking into consideration equations (A.26),(A.31) and (A.32) and repeating the algebraic argument used in Theorem A.1, it is clear that

$$R_E \approx \lambda_{11} i + B\Lambda$$

1 $\mathbb{R}_o^+$ is the set of the non-negative real numbers.

where Λ_{11} and the elements of Λ are non-zero real numbers.

Q.E.D.

The proof of Theorem A.2 includes the proof of the following conclusion.

Conclusion A.2 The "no arbitrage" condition of Ross' implies the weaker "no arbitrage" condition.

Finally by making use of the weaker "no arbitrage" condition one can easily prove the Corollaries A.1 and A.2 .

APPENDIX B

The "no arbitrage" Condition of the Capital Asset Pricing Model(Sample Risk-Return Exact Linear Relationship)

This appendix proves the existence of a "no arbitrage" condition behind the C.A.P.M.(S.R.R.E.L.R.). It also derives the C.A.P.M.(S.R.R.E.L.R.) by making use of such a "no arbitrage" condition.

Notation and Properties

Let N be a finite number of risky securities,where $N \in \{n_1 : n_1 \text{ is a finite integer}\}$. Throughout the present appendix the following notation will be used:

V, the (Nxl) covariance matrix of returns on N securities.

R, the (Nxl) column vector of mean returns.

i, the (Nxl) unit vector.

X_{M1}, a (Nxl) decision column vector of investment proportions defining an arbitrary portfolio M1.

r_{M1}, the (scalar) mean return on a portfolio M1 .

σ^2_{M1}, the(scalar) variance of return on a portfolio M1.

σ_{M1}, the (scalar) standard deviation of return on a portfolio M1.

$\sigma_{M1\ M2}$, the (scalar) covariance of returns for any pair of portfolios M1 and M2 .

The following statements are stated without proof:

Mean-variance (mean-standard deviation) B.P.[1]: $X_{M1} = V^{-1}(R\ i)A^{-1}\begin{pmatrix} r_{M1} \\ 1 \end{pmatrix}$ (B.1)

where

$$A = \begin{bmatrix} R\ V^{-1}R & R\ V^{-1}i \\ R\ V^{-1}i & i\ V^{-1}i \end{bmatrix} = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \qquad (B.2)$$

(Merton(1972),Roll(1977))

Variance of a B.P. : $\sigma_{M1}^2 = (r_{M1}i)A^{-1}\begin{pmatrix} r_{M1} \\ 1 \end{pmatrix}$ (B.3)

(Merton(1972),Roll(1977))

Covariance between the returns on two portfolios when one at least is a B.P. : $\sigma_{M1\ M2} = (r_{M1}\ 1)A^{-1}\begin{pmatrix} r_{M2} \\ 1 \end{pmatrix}$ (B.4)

(Roll(1977))

The following conclusion is required:

Conclusion B.1 All expected return-standard deviation portfolios which use a positive amount of wealth and have the same risk in a B.P. (other than the G.B.P.[2])

---

1 For the definition of a B.P. see page 50.

2 For the definition of the G.B.P. see page 85 .

measured relative to the risk of the B.P. earn the same expected return.

Proof

Without loss of generality it can be assumed that two portfolios, call them g1 and g2, have the same risk in a B.P., call it M1, measured relative to the risk of M1. That is

$$\frac{\sigma_{g1\ M1}}{\sigma_{M1}^2} = \frac{\sigma_{g2\ M1}}{\sigma_{M1}^2} \qquad (B.5)$$

The portfolios g1 and g2 are not both B.P.'s since by the constuction of the boundary portfolio set there are no two B.P.'s with the same expected return.

By simplifing equation (B.5) gives

$$\sigma_{g1\ M1} = \sigma_{g2\ M2} \qquad (B.6)$$

Hence in light of equation (B.4) equation (B.6) becomes

$$\frac{a - br_{g1} - br_{M1} + cr_{g1}r_{M1}}{ac - b^2} = \frac{a - br_{g2} - br_{M1} + cr_{g2}r_{M1}}{ac - b^2}$$

After a short manipulation the last equation yields

$$r_{g1} = r_{g2}$$

Q.E.D.

A special case of conclusion B.1 is the following:

<u>Conclusion B.2</u> All the expected return-standard deviation portfolios that use a positive amount of wealth and have zero risk in a B.P. (other than the G.B.P.) measured relative to the risk of the B.P. earn the same expected return.

The following conclusion is an application of Condition(1) given in Section 4.4 .

<u>Conclusion B.3</u> All portfolios that use a positive amount of wealth and have risk in a B.P. (other than the G.B.P.) measured relative to the risk of the B.P. equal to 1,earn the same expected return.

<u>Proof</u>

Consider two portfolios g1 and g2 such that

$$\frac{\sigma_{g1\ M1}}{\sigma_{M1}^2} = \frac{\sigma_{g2\ M2}}{\sigma_{M1}^2} = 1 \tag{B.7}$$

According to Condition(1), given in section 4.5, one has $r_{g1} = r_{g2}$. Equation (B.7) implies

$$\sigma_{g1\ M1} = \sigma_{M1}^2 \tag{B.8}$$

When equations (B.3),(B.4) and (B.8) are taken into account and obvious simplifications are made, the following equation is observed

$$(r_{M1} - r_{g1})(b - cr_{M1}) = 0$$

Since the portfolio M1 is different than the G.B.P. one can infer that $r_{M1} \neq b/c$ (where b/c is the expected return of the G.B.P.). Therefore the last equation gives $r_{M1} = r_{g1}$.

Q.E.D.

Conclusion B.2 is helpful in proving the next lemma.

LEMMA B.1 For any mean-standard deviation portfolio M1,other than the G.B.P. , the following statements are equivalent:

(i) M1 is a B.P.

(ii) There exists an exact linear relationship between the mean return vector R and the covariance vector VX_{M1}.

Proof

(i)$\Longrightarrow$(ii) Assume that M1 is a B.P., other than the G.B.P. Then in view of Conclusion B.2 all the portfolios that are uncorrelated with M1 have the same mean return. Suppose $X_{Z_{M1}}$ and $X_{\hat{Z}_{M1}}$ are two distinct (Nx1) investment proportions vectors defining two portfolios Z_{M1} and $\hat{Z}_{M2}$,respectively,where

$$X'_{Z_{M1}}(VX_{M1}) = 0 \tag{B.9}$$

$$X'_{\hat{Z}_{M1}}(VX_{M1}) = 0 \tag{B.10}$$

with

$$X'_{Z_{M1}} i = 1 \tag{B.11}$$

$$X'_{\hat{Z}_{M1}} i = 1 \tag{B.12}$$

Moreover holds

$$X'_{Z_{M1}} R = X'_{\hat{Z}_{M1}} R \tag{B.13}$$

Equations (B.9),(B.10),(B.11),(B.12) and (B.13) easily give

$$(X_{Z_{M1}} - X_{\hat{Z}_{M1}})' \frac{VX_{M1}}{\sigma_{M1}^2} = 0 \tag{B.14}$$

$$(X_{Z_{M1}} - X_{\hat{Z}_{M1}})' i = 0 \tag{B.15}$$

$$(X_{Z_{M1}} - X_{\hat{Z}_{M1}})' R = 0 \tag{B.16}$$

From equations (B.14),(B.15) and (B.16) it can be observed that the vectors VX_{M1}, i and R are orthogonal to the vector $(X_{Z_{M1}} - X_{\hat{Z}_{M1}})'$. Therefore the application of the algebraic argument used in Chapter 4 produces

$$R = \lambda_1 i + \lambda_2 \frac{VX_{M1}}{\sigma_{M1}^2} \tag{B.17}$$

where λ_1 and λ_2 are real numbers.

(ii)$\Longrightarrow$(i) To obtain a contradiction,suppose equation (B.17) implies that M1 is not a B.P. From equation (B.17) it is evident that

$$r_j = r_{Z_{M1}} + (r_{M1} - r_{Z_{M1}}) \frac{\sigma_{jM1}}{\sigma_{M1}^2} \tag{B.18}$$

where j is an individual security.

If M1 is not a B.P. , then always exists a B.P.,call it M2, such that $r_{M1} = r_{M2}$ and $\sigma_{M1}^2 > \sigma_{M2}^2$.

Since M1 is a B.P. one can prove

$$r_j = r_{Z_{M1}} + (r_{M2} - r_{Z_{M1}}) \frac{\sigma_{JM1}}{\sigma_{M2}^2} \tag{B.19}$$

Since $r_{M1} = r_{M2}$ it may be concluded that $\sigma_{jM1} = \sigma_{JM2}$.
Thus a direct comparison between equations (B.18) and (B.19) confirms that

$$\sigma_{M1}^2 = \sigma_{M2}^2$$

This is a contradiction . Hence M1 is a B.P.

Q.E.D.

APPENDIX C

Tests for the Intertemporal Stationarity of the Security Variances and Mean Returns

The present appendix contains some details of the F-test concerning the intertemporal stationarity of the security variances and a t-test regarding the intertemporal stationarity of the security mean returns.

Table C.1 An F-test for the intertemporal stationarity of the security variances and a t-test for the intertemporal stationarity of the security mean returns using two nonoverlapping subperiods. of length 60.[1]
Sample A : Period :1/1972-12/1981 , Number of securities : 672 .

NUMBER OF SECURITY	F-VALUE[2]	T-VALUE[3]	NUMBER OF SECURITY	F-VALUE	T-VALUE
1	3.9	-0.5	32	2.3	-0.3
2	1.9*	-0.4	33	1.7*	-0.7
3	2.5	-0.5	34	2.2	-1.3
4	2.0	-1.6	35	1.6*	-0.3
5	2.6	-0.9	36	5.0	-1.2
6	1.9*	-2.0	37	2.6	-0.9
7	2.3	-1.1	38	1.7*	-0.5
8	2.3	-1.8	39	1.3*	-0.8
9	2.3	-1.2	40	3.4	-1.8
10	2.8	-0.7	41	1.8*	-1.0
11	2.9	-0.7	42	1.4*	-0.5
12	1.2*	1.1	43	1.2*	-0.5
13	1.8*	-1.5	44	1.9*	0.1
14	2.5	-0.9	45	1.8*	1.7
15	1.7*	-0.5	46	3.7	-0.9
16	1.0*	0.6	47	3.5	-0.4
17	2.8	-1.0	48	2.8	-1.0
18	2.7	-0.8	49	2.8	-0.9
19	1.0*	-0.8	50	2.6	-1.0
20	2.5	-0.9	51	1.0*	-0.3
21	2.2	-0.7	52	1.7*	-0.5
22	2.4	-1.4	53	2.2	-1.2
23	1.5*	-0.5	54	1.7*	-0.1
24	1.6*	-1.8	55	4.3	-0.8
25	2.0	-1.4	56	1.5*	-1.4
26	2.1	-0.3	57	3.1	-1.4
27	2.3	-1.3	58	1.3*	-0.7
28	3.1	-0.4	59	1.1*	-1.1
29	1.6*	-0.3	60	2.6	-1.1
30	1.1*	-0.8	61	2.6	-0.1
31	1.6*	-1.7	62	2.8	-0.4

Table C.1
(Continued)

NUMBER OF SECURITY	F-VALUE	T-VALUE	NUMBER OF SECURITY	F-VALUE	T-VALUE
63	1.7*	-1.1	101	1.4*	-0.6
64	2.7	-0.4	102	1.4*	-0.8
65	3.1	-1.6	103	1.4*	-0.4
66	2.3	-1.6	104	1.0*	0.3
67	2.2	-0.9	105	2.1	-0.9
68	2.8	-1.5	106	2.4	-2.2
69	1.5*	-0.2	107	1.8*	-0.9
70	1.8*	-1.7	108	1.7*	-0.2
71	1.0*	0.8	109	1.4*	-1.2
72	2.4	-0.3	110	2.3	-1.6
73	1.2*	-0.8	111	2.4	-1.2
74	2.8	-0.9	112	2.7	-1.1
75	1.2*	-0.3	113	1.4*	0.5
76	4.2	-0.8	114	3.8	-1.1
77	2.1	-0.7	115	1.8*	-0.9
78	2.9	-0.4	116	3.4	-1.3
79	2.0	-1.1	117	1.8*	-0.8
80	1.0*	-0.2	118	2.9	-2.0
81	2.0	-0.3	119	2.4	-1.1
82	1.4*	-1.4	120	2.3	-1.3
83	1.3*	-1.4	121	2.6	-0.1
84	1.5*	-2.1	122	1.7*	-2.1
85	1.9*	-0.7	123	2.7	-0.6
86	2.4	0.4	124	1.7*	-1.3
87	1.2*	-1.2	125	1.1*	-1.4
88	1.8*	-1.7	126	1.4*	-0.3
89	1.6*	-0.7	127	2.3	-1.7
90	2.8	-0.8	128	1.6*	-2.0
91	2.5	-0.8	129	2.0	-0.8
92	1.8*	-0.4	130	2.4	-1.1
93	3.2	-0.9	131	2.7	-1.3
94	2.7	-0.7	132	1.1*	-0.8
95	2.4	-1.2	133	2.0	-0.7
96	1.6*	-0.7	134	2.9	-0.7
97	1.7*	-1.3	135	1.3*	-0.8
98	1.9*	-0.3	136	1.1*	-0.8
99	2.0	-1.8	137	2.5	-1.2
100	3.8	-0.8	138	1.0*	-0.8

Table C.1
(Continued)

NUMBER OF SECURITY	F-VALUE	T-VALUE	NUMBER OF SECURITY	F-VALUE	T-VALUE
139	2.1	-0.6	177	1.5*	-0.6
140	3.4	-2.3	178	1.8*	-1.2
141	2.6	-0.8	179	2.4	-1.7
142	1.9*	-0.2	180	2.5	-1.4
143	1.9*	-0.1	181	1.8*	-1.7
144	2.0	-1.3	182	3.7	0.2
145	1.0*	-0.5	183	3.7	-0.1
146	1.2*	-0.1	184	1.2*	-1.5
147	3.2	-1.1	185	1.3*	-0.3
148	1.8*	-0.5	186	1.8*	0.2
149	1.9*	-0.1	187	1.9*	-1.5
150	1.5*	-0.9	188	1.0*	-1.3
151	2.0	-1.1	189	3.8	-0.4
152	3.0	-0.8	190	1.3*	-0.9
153	2.7	-0.9	191	1.6*	-0.5
154	2.0	-0.4	192	2.3	-0.4
155	3.6	-1.0	193	2.1	-0.6
156	1.3*	0.7	194	2.1	-0.9
157	1.7*	-0.3	195	2.0	-0.6
158	2.4	-0.9	196	1.9*	-1.1
159	1.7*	-1.2	197	2.3	-0.3
160	1.8*	-1.1	198	2.1	-1.5
161	2.3	-1.3	199	1.7*	-0.5
162	2.8	-0.8	200	1.6*	-1.2
163	2.4	-1.3	201	1.5*	-0.2
164	1.9*	-0.4	202	1.3*	0.1
165	1.7*	-0.6	203	1.0*	-1.1
166	1.8*	-1.7	204	2.3	-1.3
167	1.1*	-0.4	205	1.9*	-0.5
168	1.4*	-1.4	206	2.8	-1.9
169	2.2	-0.9	207	2.2	-1.2
170	2.8	-0.9	208	2.9	-0.2
171	2.1	-1.7	209	3.4	-1.1
172	2.7	-0.3	210	1.8*	-1.8
173	4.4	-0.6	211	1.9*	-1.1
174	1.0*	-0.4	212	1.6*	0.4
175	2.7	-2.0	213	1.5*	-0.8
176	1.0*	-0.7	214	2.3	-0.5

Table C.1
(Continued)

NUMBER OF SECURITY	F-VALUE	T-VALUE	NUMBER OF SECURITY	F-VALUE	T-VALUE
215	2.0	-0.5	253	1.4*	-0.7
216	1.4*	0.1	254	1.8*	-0.5
217	1.9*	-0.6	255	2.9	-0.8
218	2.8	-1.0	256	2.2	-1.2
219	2.7	-0.6	257	2.7	-1.2
220	2.3	-0.5	258	3.5	-1.6
221	1.6*	-1.4	259	2.3	-0.8
222	3.0	-1.5	260	1.2*	0.6
223	3.0	-1.8	261	1.7*	-0.2
224	2.5	-1.3	262	1.9*	-0.8
225	2.9	-1.5	263	1.8*	-0.7
226	2.0	-1.2	264	1.5*	-1.5
227	1.0*	0.5	265	3.6	-0.9
228	2.2	-1.2	266	2.6	-1.7
229	1.0*	-0.4	267	2.2	-0.9
230	1.3*	-0.2	268	2.0	-1.3
231	2.8	-0.4	269	1.5*	-1.7
232	2.6	-1.3	270	1.6*	-0.9
233	1.8*	-2.1	271	2.8	-0.2
234	2.4	-2.3	272	1.9*	-1.7
235	1.6*	-0.2	273	2.0	-1.7
236	2.3	-0.5	274	1.1*	-1.0
237	1.2*	-0.4	275	3.2	-0.7
238	2.2	-0.8	276	2.2	-1.2
239	1.6*	-1.7	277	2.5	0.4
240	1.7*	0.3	278	2.2	-2.2
241	4.5	-1.0	279	7.9	-1.7
242	1.5*	-1.7	280	2.3	-0.9
243	1.1*	-0.5	281	1.6*	-1.0
244	2.8	-0.9	282	1.6*	-0.2
245	1.4*	-0.3	283	1.5*	-0.5
246	1.5*	-1.4	284	1.6*	-1.7
247	1.8*	-0.8	285	2.2	-0.5
248	4.0	-0.6	286	4.1	-0.5
249	1.1*	-1.0	287	2.7	-1.9
250	6.1	-1.5	288	1.5*	-0.9
251	2.1	-0.5	289	3.3	-1.3
252	1.7*	-0.3	290	2.2	-1.5

Table C.1
(Continued)

NUMBER OF SECURITY	F-VALUE	T-VALUE	NUMBER OF SECURITY	F-VALUE	T-VALUE
291	3.1	-0.7	329	2.6	-0.7
292	5.9	-1.1	330	1.4*	-1.8
293	3.2	-0.6	331	1.0*	-0.7
294	1.1*	-1.7	332	1.2*	-0.8
295	4.3	-0.7	333	2.4	-0.8
296	1.4*	-0.6	334	2.0	-0.6
297	4.9	-0.4	335	1.8*	-0.8
298	1.4*	0.3	336	2.3	-1.3
299	2.3	-1.0	337	2.4	-0.8
300	2.3	-0.5	338	1.3*	-1.2
301	1.7*	-1.3	339	1.6*	-1.0
302	2.2	-1.7	340	4.3	-0.6
303	2.6	0.3	341	1.6*	-1.0
304	1.0*	-1.2	342	1.8*	-0.6
305	1.2*	-0.9	343	1.0*	-1.6
306	2.0	-1.2	344	1.1*	-1.5
307	1.1*	-0.8	345	2.7	-0.4
308	1.0*	-1.1	346	2.3	-0.5
309	2.5	-1.5	347	1.2*	-1.2
310	2.1	-0.9	348	1.1*	-1.0
311	3.0	-1.1	349	2.5	-1.0
312	1.1*	-0.1	350	2.5	-1.4
313	1.2*	-0.3	351	1.6*	-1.9
314	1.1*	-1.1	352	1.9*	-1.4
315	2.6	-1.0	353	2.8	-0.6
316	2.2	-1.6	354	1.1*	-1.8
317	3.1	0.6	355	2.5	-1.0
318	1.0*	-1.6	356	2.5	-0.6
319	1.1*	0.5	357	2.1	-0.6
320	1.1*	-0.6	358	2.5	-0.8
321	2.1	-0.9	359	1.8*	-1.1
322	2.0	-0.6	360	2.4	-1.0
323	1.3*	0.1	361	1.8*	-0.5
324	1.6*	0.3	362	1.8*	-0.8
325	2.0	-0.1	363	1.3*	-0.2
326	2.1	-0.9	364	2.4	-1.0
327	3.3	-1.4	365	2.9	-0.5
328	1.1*	0.3	366	2.0	-0.6

Table C.1
(Continued)

NUMBER OF SECURITY	F-VALUE	T-VALUE	NUMBER OF SECURITY	F-VALUE	T-VALUE
367	4.0	-1.1	405	2.0	-1.7
368	4.0	-1.4	406	2.1	-1.0
369	3.9	-1.1	407	1.2*	-0.2
370	3.1	-0.7	408	2.9	-1.3
371	1.2*	-0.7	409	2.3	-0.8
372	2.6	-1.5	410	2.4	-0.6
373	2.3	-0.5	411	2.7	-1.1
374	2.0	-1.6	412	1.1*	-0.1
375	4.0	-1.1	413	2.9	-2.0
376	1.2*	-0.6	414	1.5*	0.4
377	1.8*	-1.2	415	2.0	-0.6
378	1.4*	-1.2	416	1.2*	-0.2
379	1.5*	-0.3	417	2.9	0.9
380	2.0	-0.1	418	1.0*	-1.0
381	1.1*	-0.5	419	3.4	-1.0
382	1.7*	0.6	420	2.5	-0.9
383	1.7*	-0.1	421	1.0*	-0.1
384	2.8	-1.2	422	3.7	-0.9
385	2.6	-2.5	423	1.3*	-1.1
386	4.6	-0.8	424	2.4	-0.8
387	1.4*	-0.6	425	2.3	-1.2
388	2.1	-0.1	426	2.5	-0.5
389	2.8	-0.4	427	1.2*	-1.3
390	1.1*	-1.4	428	2.1	-1.4
391	1.9*	-1.1	429	2.7	-1.7
392	3.0	-0.5	430	1.5*	-1.2
393	1.9*	-1.1	431	1.7*	0.4
394	2.0	0.2	432	1.2*	-1.5
395	2.7	-1.1	433	1.2*	-0.9
396	1.3*	-0.7	434	1.3*	-1.1
397	1.9*	-0.7	435	1.8*	-1.3
398	2.0	-1.1	436	2.8	-0.8
399	2.5	-0.7	437	1.1*	-0.1
400	1.3*	-0.4	438	1.0*	-0.3
401	1.4*	-1.4	439	1.1*	-0.2
402	2.5	-1.6	440	1.5*	-1.0
403	1.6	0.8	441	1.5*	-0.7
404	3.6	-1.2	442	2.3	-0.9

Table C.1
(Continued)

NUMBER OF SECURITY	F-VALUE	T-VALUE	NUMBER OF SECURITY	F-VALUE	T-VALUE
443	2.3	-1.0	481	2.4	-1.1
444	2.3	-0.8	482	1.6*	-1.1
445	4.8	-1.0	483	2.6	-1.2
446	2.8	-1.6	484	1.4*	-1.2
447	3.1	-0.9	485	1.7*	-0.7
448	1.5*	-1.6	486	2.6	-0.7
449	1.7*	-0.8	487	1.2*	-1.1
450	2.6	0.1	488	1.9*	-1.1
451	1.2*	0.8	489	3.4	-0.9
452	2.3	-1.6	490	1.5*	-0.7
453	1.1*	-2.0	491	2.3	-0.7
454	1.4*	-0.1	492	1.5*	-0.5
455	1.6*	-1.2	493	1.1*	0.2
456	2.3	-0.7	494	1.9*	-1.5
457	1.5*	-0.6	495	1.3*	0.6
458	1.3*	-0.4	496	1.7*	-0.2
459	3.5	-1.3	497	1.2*	-0.8
460	2.0	-1.2	498	1.7*	-1.2
461	3.1	-1.0	499	2.9	-0.5
462	2.6	-0.4	500	3.7	-0.2
463	2.1	-0.1	501	1.5*	-1.6
464	3.3	-0.6	502	1.6*	-0.6
465	2.9	-1.2	503	2.3	-1.4
466	1.5*	-0.6	504	1.1*	0.2
467	1.2*	-0.4	505	1.5*	-1.6
468	1.7*	-0.7	506	3.9	-1.2
469	3.5	-0.6	507	1.6*	-0.9
470	1.1*	-1.7	508	2.8	-0.6
471	1.6*	-2.1	509	1.2*	0.4
472	1.6*	-1.8	510	2.1	-1.0
473	2.2	-1.6	511	1.9*	-1.2
474	3.2	-0.8	512	2.0	0.1
475	1.2*	-0.5	513	1.9*	-0.9
467	2.5	-0.4	514	1.5*	-0.1
477	1.9*	-1.1	515	1.0*	0.6
478	4.3	-0.7	516	1.5*	-2.1
479	3.3	-0.6	517	3.6	0.2
480	3.4	-1.4	518	5.4	-0.3

Table C.1
(Continued)

NUMBER OF SECURITY	F-VALUE	T-VALUE	NUMBER OF SECURITY	F-VALUE	T-VALUE
519	3.2	-1.3	557	1.1*	-0.9
520	1.2*	-0.7	558	1.0*	0.1
521	1.3*	-1.1	559	2.0	-0.4
522	3.1	-1.5	560	1.3*	-0.8
523	3.3	-1.2	561	1.8*	-0.5
524	1.0*	-0.2	562	2.3	-0.7
525	1.3*	-1.3	563	1.6*	-1.7
526	1.5*	-1.8	564	2.7	-0.5
527	1.8*	-0.5	565	2.2	-1.5
528	2.1	-1.0	566	2.6	-1.1
529	4.6	-1.0	567	1.8*	-1.3
530	1.6*	0.3	568	2.9	-0.1
531	2.9	-1.4	569	2.3	-1.0
532	2.1	-1.4	570	1.2*	-1.2
533	1.3*	-0.8	571	4.9	-0.8
534	6.4	-0.3	572	5.2	-0.4
535	3.3	-1.0	573	3.6	-0.8
536	2.5	-0.7	574	1.1*	-0.4
537	3.5	-0.7	575	2.5	-0.3
538	2.3	-0.8	576	3.1	-0.1
539	2.6	-0.7	577	3.0	-0.4
540	5.0	-0.6	578	1.8*	-0.8
541	2.4	-0.7	579	2.0	-0.9
542	2.8	-0.9	580	7.0	-0.7
543	2.3	-0.6	581	5.3	-0.5
544	1.8*	-0.4	582	1.4*	1.3
545	3.3	-0.9	583	1.1*	-0.7
546	1.8*	-1.3	584	1.9*	1.1
547	3.7	-0.7	585	2.6	-1.9
548	2.3	-1.3	586	2.5	-1.4
549	1.7*	-1.3	587	1.4*	0.5
550	2.5	-0.5	588	5.1	-0.7
551	2.2	-1.0	589	2.0	-1.0
552	2.2	-0.7	590	1.0*	-1.3
553	2.4	-0.6	591	5.3	-0.8
554	1.2*	-0.8	592	2.2	-2.0
555	4.1	-0.5	593	2.2	-0.9
556	1.1*	0.4	594	3.2	0.4

Table C.1
(Continued)

NUMBER OF SECURITY	F-VALUE	T-VALUE	NUMBER OF SECURITY	F-VALUE	T-VALUE
595	2.1	-1.3	630	4.0	-1.0
596	1.7*	-1.5	631	2.1	-0.6
597	1.8*	-0.7	632	1.0*	-1.0
598	2.5	-1.0	633	2.8	-1.1
599	1.7*	0.9	634	2.0	0.1
600	1.5*	-0.2	635	1.3*	-0.4
601	2.8	-1.0	636	1.6*	-1.0
602	1.5*	-2.2	637	3.3	-1.2
603	3.3	-1.1	638	2.3	-0.6
604	3.3	-0.6	639	1.4*	-1.1
605	1.5*	-1.8	640	1.3*	-1.4
606	1.0*	-0.6	641	1.0	-1.1
607	2.9	-0.7	642	1.2*	-2.3
608	2.0	-1.1	643	1.4*	-1.7
609	1.9*	-0.2	644	1.0*	-0.8
610	2.0	-0.2	645	1.5*	-1.2
611	2.7	-1.0	646	1.0*	-1.5
612	1.0*	-0.6	647	1.4*	-0.9
613	2.8	-1.2	648	1.9*	0.3
614	2.1	-1.0	649	1.1*	-0.3
615	2.1	-1.1	650	1.5*	-0.7
616	1.3*	0.3	651	5.4	-0.7
617	2.7	-1.9	652	1.0*	-0.4
618	1.3*	-0.1	653	1.9*	-0.2
619	2.6	-1.2	654	1.4*	-1.0
620	1.2*	-1.9	655	2.5	-1.0
621	3.9	-1.4	656	3.5	-1.6
622	1.7*	-0.3	657	2.1	0.6
623	1.8*	-0.9	658	1.7*	-2.4
624	1.4*	-1.0	659	1.0*	-0.4
625	2.8	-0.1	660	1.4*	-0.5
626	2.5	-0.6	661	1.6*	-0.9
627	1.9*	-1.7	662	2.0	-0.9
628	2.1	-1.2	663	1.0*	-0.2
629	2.0	-1.1	664	1.0*	-0.3

Table C.1
(Continued)

NUMBER OF SECURITY	F-VALUE	T-VALUE	NUMBER OF SECURITY	F-VALUE	T-VALUE
665	1.3*	-0.4	669	4.1	-1.2
666	1.4*	-0.7	670	3.6	0.1
667	1.7*	0.8	671	1.1*	-0.1
668	2.3	-1.4	672	2.8	-0.1

1 Subperiods 1/1972 - 12/1976 and 1/1977-12/1981 .

2 The null hypothesis is that the security variance is intertemporally stationary . Asterisks denote that the null hypothesis is rejected at the 99% level of confidence .

3 The null hypothesis that the security mean return is intertemporally stationary is accepted in all the cases at the 99% level of confidence .

Table C.2 An F-test for the intertemporal stationarity of the security variances and a t-test for the intertemporal stationarity of the security mean returns using four nonoverlapping subperiods of length 75.
Sample B: Period:11/1956-12/1981,
Number of securities:200 .

SUBPERIODS:11/1956-1/1963 and 2/1963-4/1969					
NUMBER OF SECURITY	F-VALUE[1]	T-VALUE[2]	NUMBER OF SECURITY	F-VALUE	T-VALUE
1	1.8*	0.4	31	1.2*	-1.6
2	1.8*	0.9	32	1.4*	-0.4
3	1.2*	-0.2	33	1.5*	-0.2
4	1.2*	-0.8	34	3.0	0.6
5	1.2*	0.2	35	1.2*	0.6
6	1.1*	0.4	36	1.0*	-0.2
7	2.0	-0.3	37	2.1	0.3
8	1.4*	1.2	38	1.1*	-0.3
9	1.2*	0.7	39	1.1*	-0.1
10	1.2*	0.5	40	1.0*	1.1
11	1.6*	-0.1	41	1.5*	0.8
12	1.4*	0.7	42	1.4*	-0.5
13	1.3*	0.9	43	1.4*	0.6
14	1.4*	1.1	44	1.2*	1.7
15	1.5*	2.0	45	1.2*	1.3
16	4.4	0.2	46	1.2*	0.5
17	2.4	0.8	47	1.3*	1.6
18	1.1*	0.7	48	1.5*	1.1
19	1.1*	-0.4	49	1.3*	1.6
20	1.5*	0.5	50	1.2*	-0.5
21	1.8*	1.7	51	1.9*	0.5
22	1.3*	-0.1	52	1.4*	1.2
23	1.1*	1.4	53	2.3	-1.2
24	2.2	0.1	54	1.1*	-0.1
25	1.2*	-0.6	55	1.4*	0.6
26	1.3*	-0.1	56	3.0	0.3
27	1.3*	-0.7	57	1.2*	0.5
28	1.3*	-0.7	58	1.5*	0.1
29	1.5*	-0.1	59	1.0*	1.0
30	1.5*	0.8	60	1.4*	1.5

Table C.2
(Continued)

NUMBER OF SECURITY	F-VALUE	T-VALUE	NUMBER OF SECURITY	F-VALUE	T-VALUE
61	1.1*	0.4	98	1.3*	1.7
62	2.0	0.8	99	1.2*	1.0
63	1.9*	0.8	100	1.4*	0.9
64	1.3*	0.8	101	1.1*	0.9
65	1.0*	0.4	102	1.4*	1.3
66	1.0*	0.9	103	1.0*	-1.4
67	1.2*	1.4	104	1.3*	0.1
68	2.0	-0.4	105	1.7*	1.1
69	2.0	0.4	106	1.6*	0.3
70	2.3	1.7	107	1.0*	0.2
71	1.4*	-0.1	108	1.4*	0.5
72	1.4*	0.2	109	1.0*	1.3
73	1.1*	0.5	110	1.5*	0.2
74	1.5*	1.2	111	1.3*	-0.7
75	1.2*	0.7	112	1.4*	-0.1
76	2.1	-0.4	113	2.2	0.6
77	1.7*	-0.2	114	2.7	0.2
78	1.0	1.6	115	1.3*	-0.1
79	1.1*	-0.4	116	1.5*	0.2
80	1.1*	1.1	117	1.7*	-0.8
81	1.6*	-0.4	118	2.3	1.7
82	1.1*	1.8	119	1.5*	0.7
83	1.0*	0.5	120	1.4*	-2.0
84	1.0*	0.7	121	2.0	-0.7
85	1.1*	-0.5	122	1.9*	0.7
86	1.2*	0.3	123	1.1*	1.4
87	1.5*	0.4	124	1.1*	0.7
88	1.1*	1.1	125	1.0*	1.7
89	2.2	1.5	126	1.6*	0.1
90	1.4*	0.4	127	2.1	0.5
91	1.6*	0.4	128	1.8*	0.9
92	1.4*	0.4	129	1.4*	0.7
93	1.2*	-0.9	130	1.1*	0.9
94	1.5*	-1.0	131	1.5*	-1.5
95	1.1*	0.6	132	2.0	-0.1
96	1.0*	0.6	133	1.2*	0.2
97	1.5*	-0.1	134	1.7*	-0.4

Table C.2
(Continued)

NUMBER OF SECURITY	F-VALUE	T-VALUE	NUMBER OF SECURITY	F-VALUE	T-VALUE
135	1.0*	-0.3	172	1.5*	0.9
136	1.0*	1.0	173	1.7*	-0.2
137	1.5*	0.4	174	1.1*	0.3
138	1.5*	-0.1	175	1.1*	1.0
139	1.7*	-1.1	176	1.0*	0.9
140	1.0*	-1.5	177	2.0	0.8
141	1.0*	0.7	178	4.1	0.6
142	1.1*	0.3	179	1.0*	0.9
143	1.3*	0.5	180	1.7*	1.2
144	1.0*	-0.3	181	9.3	-0.5
145	3.3	0.9	182	1.5*	0.1
146	1.1*	1.0	183	1.6*	1.3
147	1.2*	0.8	184	1.2*	0.7
148	1.1*	-1.2	185	1.5*	-1.6
149	1.7*	-0.3	186	1.1*	0.4
150	1.2*	0.2	187	1.1*	0.4
151	1.1*	-0.1	188	1.6*	1.2
152	1.5*	-1.0	189	1.4*	0.4
153	1.2*	0.7	190	1.5*	1.8
154	1.3*	0.5	191	1.3*	0.3
155	1.6*	0.7	192	1.3*	-0.7
156	1.2*	0.6	193	1.0*	-1.7
157	1.2*	-1.0	194	1.8*	-0.4
158	1.5*	0.5	195	1.8*	1.5
159	1.6*	2.0	196	1.1*	-0.2
160	3.6	1.6	197	1.3*	-0.7
161	1.0*	-0.2	198	1.2*	-0.1
162	1.2*	1.5	199	1.7*	0.3
163	3.2	1.0	200	1.3*	1.3
164	1.0*	0.4			
165	1.1*	-0.9			
166	1.0*	0.4			
167	1.1*	-0.8			
168	1.3*	0.3			
169	1.8*	0.1			
170	1.0*	0.5			
171	1.3	1.1			

Table C.2
(Continued)

SUBPERIODS: 5/1969-7/1975 and 8/1975-10/1981					
NUMBER OF SECURITY	F-VALUE [1]	T-VALUE [2]	NUMBER OF SECURITY	F-VALUE	T-VALUE
1	2.0	-0.6	38	1.6*	-1.8
2	1.1*	0.7	39	1.1*	-2.3
3	1.7*	-0.6	40	1.3*	-1.7
4	1.8*	-0.1	41	2.0	-0.8
5	2.3	-0.6	42	1.8*	-0.9
6	1.1*	-0.4	43	1.5*	-1.4
7	1.6*	-0.9	44	3.0	-0.8
8	1.9*	-0.8	45	1.7*	-0.9
9	1.0*	-1.3	46	2.5	-0.9
10	2.4	-1.4	47	1.7*	-0.4
11	2.3	-0.4	48	2.0	-0.5
12	2.0	-0.3	49	1.8*	-1.3
13	1.3*	0.1	50	1.6*	-1.0
14	1.4*	-0.9	51	1.5*	-0.6
15	1.2*	-0.7	52	1.3*	-0.2
16	1.2*	-0.4	53	1.2*	-0.9
17	1.8*	-0.4	54	1.9*	-0.9
18	2.7	-2.2	55	1.2*	-1.2
19	1.5*	-1.3	56	1.2*	0.2
20	3.6	-0.5	57	1.8*	-2.0
21	1.6*	-0.4	58	3.5	-0.7
22	2.0	0.2	59	2.1	-0.5
23	1.7*	-0.9	60	2.2	-0.1
24	1.5*	-0.7	61	1.2*	-0.1
25	1.7*	-1.0	62	1.9*	-0.2
26	1.2*	-1.6	63	1.9*	-0.1
27	1.9*	-0.7	64	1.8*	-0.9
28	2.1	-1.1	65	1.5*	-1.4
29	1.3*	0.3	66	2.4	-1.4
30	2.4	-0.6	67	1.7*	-1.0
31	1.6*	-1.8	68	1.7*	-0.7
32	1.1*	0.9	69	5.3	-0.5
33	1.5*	0.4	70	1.0*	-0.8
34	1.5*	-0.4	71	1.6*	-0.4
35	1.8*	-1.3	72	1.5*	-0.4
36	2.3	-1.1	73	2.1	-1.2
37	1.9*	-1.2	74	1.5*	-0.9

Table C..2
(Continued)

NUMBER OF SECURITY	F-VALUE	T-VALUE	NUMBER OF SECURITY	F-VALUE	T-VALUE
75	1.0*	-0.7	112	1.1*	-2.0
76	2.6	-0.5	113	1.8*	-2.5
77	1.3*	-1.1	114	1.0*	-0.2
78	2.8	0.9	115	1.6*	-1.3
79	1.2*	-0.6	116	2.4	-0.9
80	1.2*	-0.5	117	1.5*	-0.1
81	1.3*	-0.4	118	1.0*	-1.2
82	1.3*	-1.4	119	1.0*	-0.9
83	2.4	-0.3	120	1.4*	-0.3
84	1.2*	-1.2	121	1.8*	-1.3
85	1.6*	0.4	122	1.4*	-0.5
86	1.7*	-0.2	123	2.0	-0.1
87	1.7*	-0.7	124	1.3*	-0.3
88	1.1*	-0.6	125	2.3	-0.3
89	1.4*	-0.6	126	1.0*	-1.3
90	2.1	-0.7	127	2.1	-0.6
91	1.2*	-0.1	128	1.0*	0.1
92	1.2*	-0.4	129	1.0*	-0.6
93	3.0	-0.1	130	1.8*	-0.2
94	1.6*	-0.9	131	2.4	-0.1
95	3.9	-0.8	132	1.4*	-0.7
96	1.2*	-0.8	133	1.1*	-1.6
97	1.0*	-1.2	134	1.2*	-1.8
98	2.0	-0.6	135	1.2*	0.4
99	1.1*	0.3	136	1.6*	-0.6
100	1.2*	0.8	137	1.9*	-0.6
101	1.1*	-1.1	138	1.5*	-0.8
102	1.4*	-0.4	139	1.8*	-0.3
103	1.5*	-1.5	140	1.1*	-1.9
104	1.2*	-0.3	141	1.1*	-1.9
105	1.0*	0.4	142	2.2	-0.3
106	1.2*	-0.3	143	1.4*	-0.9
107	1.7*	0.1	144	1.3*	-1.1
108	1.8*	-0.6	145	1.6*	-0.8
109	1.7*	-1.6	146	2.6	-0.7
110	2.3	-0.7	147	1.4*	-0.3
111	2.1	-1.3	148	1.5*	2.1

Table C.2
(Continued)

NUMBER OF SECURITY	F-VALUE	T-VALUE	NUMBER OF SECURITY	F-VALUE	T-VALUE
149	1.2*	-1.2	175	1.2*	-0.9
150	1.0*	0.6	176	1.6*	-0.6
151	1.7*	-2.4	177	1.2*	-0.3
152	1.6	0.3	178	1.4*	-1.3
153	2.8	-1.5	179	2.0	-0.2
154	1.0*	0.5	180	2.3	-0.5
155	1.2*	-0.8	181	2.4	-1.2
156	1.8*	-0.8	182	1.3*	0.3
157	1.7*	0.3	183	2.2	-1.4
158	1.8*	-0.8	184	1.3*	-0.2
159	3.0	0.2	185	1.2*	-1.1
160	1.3*	-0.4	186	3.3	-0.7
161	1.9*	0.7	187	1.9*	-0.3
162	1.0*	-0.8	188	2.4	-0.3
163	1.0*	-0.6	189	1.6*	-0.3
164	1.0*	0.3	190	1.2*	0.4
165	2.0	-1.3	191	1.9*	-0.6
166	1.7*	-0.9	192	1.7*	-0.3
167	3.3	-0.2	193	1.3*	-0.6
168	2.1	-0.7	194	1.5*	0.1
169	1.6*	-0.8	195	1.3*	-0.4
170	1.0*	1.6	196	1.2*	-0.7
171	1.5*	-0.2	197	1.3*	-1.7
172	2.1	-0.9	198	1.0*	-0.4
173	1.9*	-0.9	199	1.2*	-0.9
174	1.3*	0.3	200	1.1	-0.6

1 The null hypothesis is that the security variance is intertemporally stationary. Asterisks denote that the null hypothesis is rejected at the 99% level of confidence.

2 The null hypothesis that the security mean return is intertemporally stationary is accepted in all the cases at the 99% level of confidence.

Table C.3 An F-test for the intertemporal stationarity of the security variances and a t-test for the intertemporal stationarity of the security mean returns using two nonoverlapping subperiods of length 151. [1]
Sample B: Period: 11/1956-12/1981,
Number of securities:200 .

NUMBER OF SECURITY	F-VALUE[2]	T-VALUE[3]	NUMBER OF SECURITY	F-VALUE	T-VALUE
1	1.7*	0.7	34	2.0	-0.2
2	2.3	-0.6	35	1.5*	1.7
3	1.8*	0.9	36	1.4*	0.6
4	2.1	0.9	37	2.9	0.1
5	2.3	0.3	38	3.0	0.1
6	1.9*	-0.1	39	4.8	1.4
7	2.8	-0.1	40	1.9*	0.3
8	2.1	0.6	41	1.3*	0.5
9	2.7	0.9	42	2.3	0.6
10	5.1	0.5	43	2.3	0.4
11	2.8	-0.1	44	3.1	0.1
12	2.1	0.1	45	2.2	-0.1
13	1.3*	1.1	46	1.7*	1.1
14	2.4	-0.3	47	2.3	1.3
15	1.7*	0.8	48	1.6*	0.6
16	1.8*	0.1	49	3.0	0.3
17	3.5	0.1	50	2.5	1.1
18	2.5	-0.5	51	1.9*	0.7
19	1.7*	0.9	52	1.6*	0.7
20	4.4	-0.9	53	2.6	0.7
21	1.9*	-0.7	54	2.1	1.0
22	1.9*	0.1	55	2.3	0.7
23	2.2	0.9	56	1.7*	1.2
24	2.4	0.3	57	2.0	-0.7
25	2.2	-0.4	58	2.5	0.3
26	3.9	-0.3	59	1.4*	-0.7
27	1.6*	-0.9	60	2.6	0.6
28	2.0	1.3	61	1.9*	0.9
29	1.3*	0.2	62	1.3*	0.8
30	1.6*	1.0	63	1.5*	0.5
31	1.7*	0.4	64	3.2	0.2
32	2.8	0.3	65	1.9*	-0.1
33	1.0*	-0.6	66	3.7	0.7

Table C.3
(Continued)

NUMBER OF SECURITY	F-VALUE	T-VALUE	NUMBER OF SECURITY	F-VALUE	T-VALUE
67	2.5	-0.2	104	1.0*	0.3
68	1.0*	1.1	105	1.9*	0.4
69	1.3*	-0.1	106	2.1	0.8
70	1.7*	0.9	107	2.6	-0.1
71	1.3*	0.9	108	1.7*	0.4
72	2.4	1.5	109	2.6	0.1
73	1.5*	0.5	110	2.0	0.8
74	2.2	1.9	111	1.5*	-0.3
75	2.2	1.0	112	1.9*	0.2
76	5.4	0.9	113	1.0*	1.3
77	1.4*	0.8	114	1.0*	0.2
78	2.6	0.9	115	1.8*	0.7
79	1.7*	-0.6	116	1.9*	0.8
80	3.7	0.1	117	1.0*	-1.6
81	2.5	0.7	118	1.9*	0.3
82	1.2*	-0.1	119	1.7*	0.3
83	2.1	1.0	120	1.0*	1.4
84	2.3	2.1	121	2.7	0.4
85	1.7*	-0.1	122	2.0	-0.3
86	1.4*	1.1	123	2.2	0.6
87	1.7*	0.6	124	1.1*	1.0
88	2.3	0.8	125	3.2	0.9
89	2.3	0.4	126	2.1	0.4
90	1.8*	-0.6	127	2.0	1.5
91	2.4	1.0	128	2.2	0.7
92	2.1	0.6	129	3.4	0.5
93	3.4	-0.3	130	2.5	0.2
94	2.2	-0.7	131	1.5*	0.4
95	2.5	-0.8	132	1.9*	0.3
96	2.1	0.7	133	1.6*	-0.2
97	1.2*	0.6	134	3.6	0.5
98	1.4*	0.2	135	2.2	0.5
99	3.6	0.9	136	2.2	1.9
100	2.2	1.2	137	2.4	0.5
101	3.6	0.4	138	1.7*	0.2
102	2.2	-0.5	139	1.8*	-0.1
103	3.9	-0.2	140	1.7*	0.8

Table C.3
(Continued)

NUMBER OF SECURITY	F-VALUE	T-VALUE	NUMBER OF SECURITY	F-VALUE	T-VALUE
141	1.7*	0.5	171	1.9*	-0.8
142	1.4*	1.2	172	1.4*	-0.1
143	1.8*	0.5	173	2.5	-0.7
144	2.0	-0.2	174	2.8	-0.1
145	1.2*	1.5	175	1.9*	1.1
146	2.5	0.2	176	2.1	0.5
147	2.5	0.2	177	2.1	1.6
148	2.6	-0.7	178	2.8	0.3
149	2.2	0.3	179	2.2	0.5
150	2.7	1.1	180	1.4*	1.0
151	4.7	-0.3	181	1.3*	0.7
152	1.0*	-0.6	182	1.9*	0.8
153	1.3*	1.3	183	2.4	-1.1
154	2.2	-0.1	184	2.5	0.5
155	1.4*	1.3	185	1.3*	-0.7
156	2.0	0.8	186	2.8	-0.2
157	1.8*	0.1	187	1.5*	0.5
158	1.9*	0.8	188	3.4	0.2
159	2.4	1.3	189	1.7*	0.4
160	1.1*	0.2	190	2.4	0.5
161	2.3	1.0	191	1.5*	0.7
162	1.0	0.3	192	2.4	-0.3
163	1.7	0.7	193	1.6*	0.8
164	2.1	0.4	194	1.2*	-0.2
165	2.3	0.3	195	2.1	0.3
166	2.3	0.6	196	2.8	0.5
167	2.8	0.3	197	3.0	-0.3
168	1.6*	0.4	198	1.4*	0.3
169	1.0	-0.6	199	1.4*	0.3
170	2.8	0.9	200	2.5	0.2

1 Subperiods 11/1956-5/1969 and 6/1969-12/1981.

2 The null hypothesis is that the security variance is intertemporally statioanary. Asterisks denote that the null hypothesis is rejected at the 99% level of confidence.

3 The null hypothesis that the security mean return is intertemporally stationary is accepted in all the cases at the 99% level of confidence.

APPENDIX D

Test for the Relationship between the Number of Factors and the Group Size

This appendix presents in details the results of the test concerning the empirical verification of the assumption that the number of factors determining the security returns remains the same across various groups of different sizes and across various groups of the same size.

Table D.1 A chi-square test indicating the adequancy of the correlation matrix for factor analysis. Sample A: Period:1/1972-12/1981, Number of securities:672 .

GROUP	GROUPS CONTAINING 5 SECURITIES DEGREES OF FREEDOM:10 CRITICAL VALUE:23.2 [1]	GROUPS CONTAINING 2 SECURITIES DEGREES OF FREEDOM:210 CRITICAL VALUE:260.6	GROUPS CONTAINING 4 SECURITIES DEGREES OF FREEDOM:861 CRITICAL VALUE:960.4
1	268.6	1287.0	4755.6
2	119.5	1062.3	4317.7
3	237.9	981.9	3826.8
4	146.2	881.8	4377.9
5	145.6	816.6	3862.3
6	158.7	1028.5	3866.7
7	318.4	1429.8	4083.0
8	128.7	880.3	3886.6
9	178.4	730.0	3828.7
10	141.5	752.8	3729.2
11	230.7	825.1	4005.0
12	329.7	1106.4	4057.5
13	314.9	980.0	4983.8
14	427.9	1327.8	4665.7
15	412.0	1114.4	4657.3
16	194.4	526.3	2913.4

1 The null hypothesis that the correlation matrix of security returns is equal to the identity matrix is rejected in all the cases at the 99% level of confidence.

Table D.2 A chi-square test indicating the adequancy of the correlation matrix for factor analysis. Sample B: Period: 11/1956-12/1981, Number of securities: 200 .

	GROUPS CONTAINING 5 SECURITIES	GROUPS CONTAINING 2 SECURITIES	GROUPS CONTAINING 4 SECURITIES
	DEGREES OF FREEDOM:10	DEGREES OF FREEDOM:190	DEGREES OF FREEDOM: 780
GROUP	CRITICAL VALUE:23.2 [1]	CRITICAL VALUE:238.2	CRITICAL VALUE: 846.0
1	584.1	1945.9	6372.5
2	329.0	1351.3	4972.7
3	290.3	1397.8	5214.7
4	235.7	1189.2	5113.5
5	270.1	1398.1	5615.4

1 The null hypothesis that the correlation matrix of security returns is equal to the identity matrix is rejected in all the cases at the 99% level of confidence.

Table D.3 A chi-square test indicating the relationship between the appropriate number of common factors and the group size.
Sample A:Period:1/1972-12/1981,Number of securities:672 .

NUMBER OF MASTER GROUP	GROUP SIZE	NUMBER OF FACTORS	x^2	DEGREES OF FREEDOM	CRITICAL VALUE FOR THE x^2 DISTRIBUTION AT THE 99% LEVEL OF CONFIDENCE
1	5	1	3.2	5	15.0
	10	3	24.6	18	34.8
	15	6	48.8	30	50.8
	21	6	122.7	99	134.6
	26	7	198.1	164	209.0
	31	7	321.5	269	325.8
	36	10	357.2	315	376.3
	42	13	440.5	393	461.1
2	5	1	11.9	5	15.0
	10	2	43.1	26	45.6
	15	2	99.8	76	107.5
	21	4	164.3	132	172.7
	26	6	221.5	184	231.5
	31	9	265.1	222	273.9
	36	12	310.3	264	320.3
	42	15	382.9	336	399.2
3	5	1	3.5	5	15.0
	10	1	39.9	35	57.3
	15	1	113.7	90	124.1
	21	3	179.8	150	193.2
	26	6	219.5	184	231.5
	31	8	289.5	245	299.4
	36	11	337.3	289	347.8
	42	13	490.9	423	493.5

Table D.3
(Continued)

NUMBER OF MASTER GROUP	GROUP SIZE	NUMBER OF FACTORS	x^2	DEGREES OF FREEDOM	CRITICAL VALUE FOR THE X^2 DISTRIBUTION AT THE 99% LEVEL OF CONFIDENCE
4	5	1	11.5	5	15.0
	10	1	40.5	35	57.3
	15	2	103.9	76	107.5
	21	5	148.3	115	153.2
	26	8	175.3	145	187.5
	31	12	199.9	159	203.4
	36	15	238.8	195	243.8
	42	16	350.2	309	369.7
5	5	1	9.5	5	15.0
	10	2	31.4	26	45.6
	15	3	75.9	63	92.0
	21	3	173.9	150	193.2
	26	6	212.9	184	231.5
	31	6	345.7	294	353.3
	36	8	424.3	370	436.2
	42	11	519.8	454	527.0
6	5	1	3.6	5	15.0
	10	1	43.0	35	57.3
	15	4	68.6	51	77.3
	21	4	126.0	132	172.7
	26	6	212.9	184	231.5
	31	10	241.5	200	255.2
	36	12	314.2	264	320.3
	42	14	421.9	364	429.7

Table D.3
(Continued)

NUMBER OF MASTER GROUP	GROUP SIZE	NUMBER OF FACTORS	x^2	DEGREES OF FREEDOM	CRITICAL VALUE FOR THE x^2 DISTRIBUTION AT THE 99% LEVEL OF CONFIDENCE
7	5	2	1.1	1	6.6
	10	2	42.3	26	45.6
	15	3	79.6	63	92.0
	21	3	183.5	150	193.2
	26	6	216.8	184	231.5
	31	9	272.1	222	275.0
	36	11	330.4	289	347.8
	42	15	387.4	336	399.2
8	5	1	1.7	5	15.0
	10	1	38.6	35	57.3
	15	1	113.0	90	124.1
	21	3	176.1	150	193.2
	26	5	243.0	205	255.0
	31	6	347.3	294	353.3
	36	8	424.8	370	436.2
	42	11	518.1	454	527.0
9	5	1	11.4	5	15.0
	10	1	33.9	35	57.3
	15	1	102.4	90	124.1
	21	3	170.1	150	193.2
	26	4	271.6	227	279.4
	31	8	282.6	245	299.4
	36	11	340.5	289	347.8
	42	15	385.0	336	399.2

Table D.3
(Continued)

NUMBER OF MASTER GROUP	GROUP SIZE	NUMBER OF FACTORS	x^2	DEGREES OF FREEDOM	CRITICAL VALUE FOR THE x^2 DISTRIBUTION AT THE 99% LEVEL OF CONFIDENCE
10	5	1	1.1	5	15.0
	10	1	37.7	35	57.3
	15	1	100.5	90	124.1
	21	3	190.2	150	193.2
	26	5	252.0	205	255.0
	31	7	318.5	269	325.8
	36	9	400.1	342	405.7
	42	12	484.3	423	493.5
11	5	1	4.3	5	15.0
	10	1	50.1	35	57.3
	15	2	93.5	76	107.5
	21	3	183.7	150	193.2
	26	5	250.0	205	255.0
	31	7	317.9	269	325.8
	36	8	420.4	370	436.2
	42	12	481.3	423	493.5
12	5	1	2.2	5	15.0
	10	2	32.7	26	45.6
	15	2	81.6	76	107.5
	21	3	184.3	150	193.2
	26	7	195.0	164	209.0
	31	7	323.3	269	325.8
	36	8	432.8	370	436.2
	42	11	512.5	454	527.0

Table D.3
(Continued)

NUMBER OF MASTER GROUP	GROUP SIZE	NUMBER OF FACTORS	x^2	DEGREES OF FREEDOM	CRITICAL VALUE FOR THE X^2 DISTRIBUTION AT THE 99% LEVEL OF CONFIDENCE
13	5	1	4.7	5	15.0
	10	2	25.2	26	45.6
	15	3	78.9	63	92.0
	21	4	162.2	132	172.7
	26	7	199.5	164	209.0
	31	7	314.0	269	325.8
	36	7	459.1	399	467.6
	42	11	510.4	454	527.0
14	5	1	12.3	5	15.0
	10	2	73.4	35	45.6
	15	3	81.4	63	92.0
	21	3	173.2	150	193.2
	26	3	283.8	250	304.9
	31	4	408.7	347	411.2
	36	6	486.7	429	500.0
	42	9	581.6	519	596.8
15	5	1	5.7	5	15.0
	10	1	42.4	35	57.3
	15	2	104.8	76	107.5
	21	4	164.5	132	172.7
	26	6	222.9	184	231.5
	31	9	263.2	222	275.0
	36	12	313.9	264	320.3
	42	15	381.4	336	399.2

Table D.3
(Continued)

NUMBER OF MASTER GROUP	GROUP SIZE	NUMBER OF FACTORS	x^2	DEGREES OF FREEDOM	CRITICAL VALUE FOR THE X^2 DISTRIBUTION AT THE 99% LEVEL OF CONFIDENCE
16	5	1	9.7	5	15.0
	10	1	46.9	35	57.3
	15	1	102.9	90	124.1
	21	4	169.7	132	172.7
	26	4	259.2	227	279.4
	31	4	398.7	347	411.2
	36	6	470.1	429	500.0
	42	8	621.8	553	633.3

Table D.4 A chi-square test indicating the relationship between the appropriate number of common factors and the group size . Sample B:Period:11/1956-12/1981,Number of Securities:200 .

NUMBER OF MASTER GROUP	GROUP SIZE	NUMBER OF FACTORS	x^2	DEGREES OF FREEDOM	CRITICAL VALUE FOR THE x^2 DISTRIBUTION AT THE 99% LEVEL OF CONFIDENCE
1	5	1	12.5	5	15.0
	10	2	42.4	26	45.6
	15	4	68.0	51	77.3
	20	4	152.6	116	158.9
	25	6	197.0	165	210.1
	30	7	293.1	246	300.5
	35	8	395.4	343	406.8
	40	10	484.7	425	495.7
2	5	1	6.2	5	15.0
	10	1	36.1	35	57.3
	15	2	86.4	76	107.5
	20	3	165.6	133	173.8
	25	5	220.7	185	232.6
	30	7	288.7	246	300.5
	35	8	393.3	343	406.8
	40	9	520.0	456	529.1
3	5	1	8.6	5	15.0
	10	2	36.1	26	45.6
	15	4	61.9	51	77.3
	20	4	149.5	116	154.3
	25	4	234.1	206	255.0
	30	6	304.3	270	326.9
	35	7	424.4	371	437.3
	40	10	483.7	425	495.7

Table D.4
(Continued)

NUMBER OF MASTER GROUP	GROUP SIZE	NUMBER OF FACTORS	x^2	DEGREES OF FREEDOM	CRITICAL VALUE FOR THE X^2 DISTRIBUTION AT THE 99% LEVEL OF CONFIDENCE
4	5	1	5.5	5	15.0
	10	2	38.0	26	45.6
	15	3	82.3	63	92.0
	20	5	122.6	100	135.8
	25	6	204.2	165	210.1
	30	6	304.3	270	326.9
	35	8	384.4	343	406.8
	40	8	544.6	488	563.6
5	5	2	1.9	1	6.6
	10	2	39.2	26	45.6
	15	3	78.7	63	92.0
	20	3	161.0	133	173.8
	25	4	249.5	206	255.0
	30	4	367.8	321	382.9
	35	5	493.4	430	501.1
	40	7	586.2	521	599.0

A P P E N D I X E

Test for the Intertemporal Stability of the Number of Factors

The present appendix gives in details the results of the test regarding the empirical verification of the assumption that the number of factors affecting the security returns remains unchanged across various time periods for the same group of securities and for different security groups.

Table E.1 A chi-square test indicating the adequancy of the correlation matrix for factor analysis.
Sample A:two nonoverlapping subperiods of length 60.

	GROUPS CONTAINING 10 SECURITIES	GROUPS CONTAINING SECURITIES	GROUPS CONTAINING SECURITIES
	DEGREES OF FREEDOM:45	DEGREES OF FREEDOM:210	DEGREES OF FREEDOM:861
SUBPERIOD	CRITICAL VALUE:69.9 [1]	CRITICAL VALUE:260.6	CRITICAL VALUE:930.3
1/1972 - 12/1976	323.3	593.5	1993.1
	310.8	555.8	1648.2
	225.0	553.0	1630.7
	148.3	415.2	1601.3
	247.5	376.4	1593.4
	210.7	467.6	1880.2
	342.0	626.0	1630.3
	276.2	437.6	1602.5
	97.4	281.2	1514.7
	79.5	397.5	1781.9
	161.1	423.3	1510.1
	274.6	544.5	1731.5
	223.3	524.5	1521.7
	405.5	658.4	2014.3
	269.0	513.5	1917.8
	134.3	412.8	1431.4
1/1977 - 1231981	185.1	503.2	1787.0
	104.2	364.3	1582.6
	81.1	352.3	1386.7
	88.5	379.2	1617.6
	134.1	388.9	1522.5
	106.7	444.7	1585.6
	262.8	578.2	1826.3
	92.4	360.5	1541.6
	101.1	365.5	1662.1
	120.4	389.3	1581.4
	109.1	326.3	1448.3
	99.3	450.3	1682.8
	91.9	342.7	1490.1
	332.1	347.7	1830.8
	183.8	529.2	1788.3
	120.2	241.6	1241.7

1 The null hypothesis that the correlation matrix of security returns is equal to the identity matrix is rejected in all the cases at the 99% level of confidence.

Table E.2 A chi-square test indicating the adequancy of the correlation matrix for factor analysis.
Sample B : three nonoverlapping subperiods of length 100 .

SUBPERIOD	GROUPS CONTAINING 10 SECURITIES DEGREES OF FREEDOM:45 CRITICAL VALUE:69.9 [1]	GROUPS CONTAINING SECURITIES DEGREES OF FREEDOM:190 CRITICAL VALUE: 238.2	GROUPS CONTAINING SECURITIES DEGREES OF FREEDOM:780 CRITICAL VALUE: 846.0
11/1956 - 2/1965	147.2	466.0	1974.6
	269.6	312.1	1461.9
	114.2	338.0	1681.8
	219.7	384.8	1476.5
	220.2	272.0	1616.8
3/1965 - 6/1973	214.5	519.2	2210.7
	278.3	531.8	1761.5
	157.9	387.3	1710.8
	269.9	398.3	1859.4
	248.7	386.2	1808.1
7/1973-10/1981	290.3	1333.0	3251.4
	350.4	1030.7	2905.9
	278.5	929.1	2745.2
	337.6	800.4	2814.7
	294.7	939.9	3121.1

1 The null hypothesis that the correlation matrix of security returns is equal to the identity matrix is rejected in all the cases at the 99% level of confidence.

Table E.3 A chi-square test indicating the adequancy of the correlation matrix for factor analysis.
Sample B : two nonoverlapping subperiods of length 151 .

	GROUPS CONTAINING 10 SECURITIES	GROUPS CONTAINING SECURITIES	GROUPS CONTAINING SECURITIES
	DEGREES OF FREEDOM:45	DEGREES OF FREEDOM:190	DEGREES OF FREEDOM:780
SUBPERIOD	CRITICAL VALUE:69.9 [1]	CRITICAL VALUE:238.2	CRITICAL VALUE:846.0
11/1956 - 5/1969	214.7	586.7	2582.9
	149.3	446.8	1850.8
	160.8	484.8	2216.2
	125.7	315.6	2017.3
	137.3	367.2	2139.1
6/1969 - 12/1981	631.2	1392.2	4271.2
	372.5	1059.3	3524.4
	252.7	1104.5	3463.2
	154.3	945.3	3502.2
	180.6	1104.4	3839.6

1 The null hypothesis that the correlation matrix of security returns is equal to the identity matrix is rejected in all the cases at the 99% level of confidence.

Table E.4 A chi-square test indicating the intertemporal stability of the number of factors. Sample A: subgroups of size 10 and two nonoverlapping subperiods of length 60.

SUBPERIOD	NUMBER OF FACTORS	x^2	DEGREES OF FREEDOM	CRITICAL VALUES FOR THE x^2 DISTRIBUTION AT THE 99% LEVEL OF CONFIDENCE
1/1972-12/1976	1	50.6	35	57.3
	2	42.7	26	45.6
	1	38.0	35	57.3
	1	41.7	35	57.3
	2	37.3	26	45.6
	1	30.3	35	57.3
	1	49.6	35	57.3
	1	49.8	35	57.3
	1	50.9	35	57.3
	1	42.4	35	57.3
	1	46.2	35	57.3
	2	37.7	26	45.6
	2	28.2	26	45.6
	2	40.4	26	45.6
	1	45.3	35	57.3
	1	46.5	35	57.3
1/1977-12/1981	2	32.3	26	45.6
	2	32.5	26	45.6
	1	46.9	35	57.3
	1	42.6	35	57.3
	1	32.8	35	57.3
	2	48.7	26	45.6
	1	35.1	35	57.3
	2	35.2	26	45.6
	1	41.0	35	57.3
	1	35.8	35	57.3
	1	33.5	35	57.3
	1	40.0	35	57.3
	1	50.8	35	57.3
	1	42.3	35	57.3
	1	34.8	35	57.3
	1	38.0	35	57.3

Table E.5 A chi-square test indicating the intertemporal stability of the number of factors.
Sample A: subgroups of size 21 and two nonoverlapping subperiods of length 60 .

SUBPERIOD	NUMBER OF FACTORS	x^2	DEGREES OF FREEDOM	CRITICAL VALUES FOR THE x^2 DISTRIBUTION AT THE 99% LEVEL OF CONFIDENCE
1/1972-12/1976	5	149.7	115	154.3
	5	141.6	115	154.3
	5	147.8	115	154.3
	8	95.7	70	100.4
	4	169.9	132	172.7
	8	96.6	70	100.4
	4	164.8	132	172.7
	5	147.1	115	154.3
	4	156.7	132	172.7
	3	170.3	150	193.2
	5	149.2	115	154.3
	5	145.4	115	154.3
	5	141.1	115	154.3
	4	160.1	132	172.7
	4	159.7	132	172.7
	5	149.2	115	154.3
1/1977-12/1981	3	179.9	150	193.2
	3	175.1	150	193.2
	3	190.5	150	193.2
	7	93.6	84	117.0
	3	185.6	150	193.2
	2	208.9	169	214.6
	3	177.4	150	193.2
	4	161.3	132	172.7
	3	179.7	150	193.2
	3	186.9	150	193.2
	3	162.5	150	193.2
	4	166.5	132	172.7
	3	178.6	150	193.2
	3	164.6	150	193.2
	4	168.1	132	172.7
	8	97.4	70	100.4

Table E.6 A chi-square test indicating the intertemporal stability of the number of factors.
Sample B : subgroups of size 10 and three nonoverlapping subperiods of length 100 .

SUBPERIOD	NUMBER OF FACTORS	x^2	DEGREES OF FREEDOM	CRITICAL VALUES FOR THE x^2 DISTRIBUTION AT THE 99% LEVEL OF CONFIDENCE
11/1956-2/1965	1	22.5	35	57.3
	1	42.6	35	57.3
	1	28.6	35	57.3
	2	39.2	26	45.6
	2	40.3	26	45.6
3/1965-6/1973	2	39.5	26	45.6
	1	50.1	35	57.3
	2	35.6	26	45.6
	1	38.1	35	57.3
	1	42.4	35	57.3
7/1973-10/1981	2	30.2	26	45.6
	1	42.4	35	57.3
	2	41.9	35	45.6
	2	25.0	26	45.6
	2	37.6	26	45.6

Table E.7 A chi-square test indicating the intertemporal stability of the number of factors. Sample B: subgroups of size 20 and three nonoverlapping subperiods of length 100 .

SUBPERIOD	NUMBER OF FACTORS	x^2	DEGREES OF FREEDOM	CRITICAL VALUES FOR THE x^2 DISTRIBUTION AT THE 99% LEVEL OF CONFIDENCE
11/1956-2/1965	3	145.8	133	174.9
	4	132.9	116	154.3
	2	166.4	151	194.3
	2	184.3	151	194.3
	2	190.7	151	194.3
3/1965-6/1973	2	176.0	151	194.3
	3	170.8	133	174.9
	3	154.5	133	174,9
	3	159.8	133	174.9
	2	178.4	151	194.3
7/1973-10/1981	4	147.1	116	154.3
	2	181.3	151	194.3
	4	142.1	116	154.3
	2	175.5	151	194.3
	3	159.4	133	174.3

Table E.8 A chi-square test indicating the intertemporal stability of the number of factors.
Sample B : subgroups of size 40 and three nonoverlapping subperiods of length 100 .

SUBPERIOD	NUMBER OF FACTORS	x^2	DEGREES OF FREEDOM	CRITICAL VALUES FOR THE x^2 DISTRIBUTION AT THE 99% LEVEL OF CONFIDENCE
11/1956-2/1965	9	517.1	456	529.1
	9	521.3	456	529.1
	7	591.4	521	599.0
	8	555.1	488	563.6
	7	587.3	521	599.0
3/1965-6/1973	10	483.1	425	495.7
	10	490.6	425	495.7
	9	503.4	456	529.1
	7	587.4	521	599.0
	6	626.5	555	635.4
7/1973-10/1981	9	522.3	456	529.1
	12	422.9	366	431.8
	10	475.5	425	495.7
	8	558.9	488	563.6
	7	584.8	521	599.0

Table E.9 A chi-square test indicating the intertemporal stability of the number of factors.
Sample B: subgroups of size 10 and two nonoverlapping subperiods of length 151.

SUBPERIOD	NUMBER OF FACTORS	x^2	DEGREES OF FREEDOM	CRITICAL VALUES FOR THE x^2 DISTRIBUTION AT THE 99% LEVEL OF CONFIDENCE
11/1956-5/1969	1	44.5	35	57.3
	1	41.7	35	57.3
	2	38.1	26	45.6
	1	42.6	35	57.3
	1	39.4	35	57.3
6/1969-12/1981	2	33.9	26	45.6
	1	40.6	35	57.3
	2	38.5	26	45.6
	1	49.6	35	57.3
	2	45.5	26	45.6

Table E.10 A chi-square test indicating the intertemporal stability of the number of factors.
Sample B: subgroups of size 20 and two nonoverlapping subperiods of length 151.

SUBPERIOD	NUMBER OF FACTORS	x^2	DEGREES OF FREEDOM	CRITICAL VALUES FOR THE x^2 DISTRIBUTION AT THE 99% LEVEL OF CONFIDENCE
11/1956-5/1969	3	150.0	133	174.9
	5	121.4	100	135.8
	3	139.0	133	174.9
	4	144.1	116	154.3
	2	163.9	151	194.3
6/1969-12/1981	4	152.5	116	154.3
	4	147.6	116	154.3
	4	150.8	116	154.3
	2	181.1	151	194.3
	3	168.9	133	174.9

Table E.11 A chi-square test indicating the intertemporal stability of the number of factors. Sample B: subgroups of size 40 and two nonoverlapping subperiods of length 151.

SUBPERIOD	NUMBER OF FACTORS	x^2	DEGREES OF FREEDOM	CRITICAL VALUES FOR THE x^2 DISTRIBUTION AT THE 99% LEVEL OF CONFIDENCE
11/1956-5/1969	9	509.5	456	529.1
	7	570.4	521	599.0
	7	569.1	521	599.0
	6	624.9	555	635.4
	6	647.2	555	635.4
6/1969-12/1981	10	488.2	425	495.7
	9	518.1	456	529.1
	11	450.9	395	463.3
	8	551.1	488	563.6
	7	586.2	521	599.0

REFERENCES

Journal titles are abbreviated as follows :

A.E.R.	=	American Economic Review.
B.	=	Biometrica.
B.J.P.S.S.	=	British Journal of Psychology, Statistical Section.
E.	=	Econometrica.
F.A.J	=	Financial Analyst Journal.
F.M.	=	Financial Management.
H.B.R.	=	Harvard Business Review.
J.A.S.A.	=	Journal of American Statistical Association.
J.B.	=	Journal of Business.
J.B.F.	=	Journal of Business Finance.
J.B.F.A.	=	Journal of Business Finance and Accounting.
J.E.T.	=	Journal of Economic Theory.
J.F.	=	Journal of Finance.
J.F.E.	=	Journal of Financial Economics.
J.F.Q.A.	=	Journal of Financial and Quantitative Analysis.
J.M.F.A.	=	Journal of the Midwest Finance Association.
J.P.E.	=	Journal of Political Economy.
J.R.S.S.	=	Journal of the Royal Statistical Society.
M.S.	=	Management Sciences.
P.	=	Psychometrica.
Q.R.E.B.	=	Quarterly Review of Economics and Business.
R.E.S.	=	Review of Economic and Statistics.

Aber, J.W. (1976) "Industry Effects and Multivariate Stock Price Behaviour," J.F.Q.A., Vol. XI, p.p. 617 - 624.

Alexander, G.J. (1977) "Mixed Security Testing for Alternative Portfolio Selection Models," J.F.Q.A., Vol. XII, pp. 817 - 832.

——— (1978) "A Reevaluation of Alternative Portfolio Selection Models Applied to Common Stocks," J.F.Q.A., Vol. XIII, pp. 71 - 78.

Alexander, G.J. and Benson, P.G. (1982) "More on Beta as a Random Coefficient," J.F.Q.A., Vol. XVII, pp. 27 - 36.

Alexander, G.J. and Chervany, N.L. (1980) "On the Estimation and Stability of Beta," J.F.Q.A., Vol.XV, pp.123 - 137.

Altman, E.I., Jacyuillat, B. and Lavasseaur, M. (1974) "Companies Analysis of Risk Measures : France and the United States," J.F., Vol. XXIX, pp. 1495 - 1511.

Appleyard, A.R., Strong, N. and Walker, M. (1982) "Mutual Fund Performance in the Context of Models of Equilibrium Capital Asset Pricing," J.B.F.A., Vol.9, pp.289 - 295.

Babcock, G.C. (1972) "A Note on Justifying Beta as a Measure of Risk," J.F., Vol. XXVII, pp. 699- 702.

Baesel, S.B. (1974) "On the Assessment of Risk : Some Further Considerations," J.F., Vol. XXIX, pp. 1491 - 1498.

Barry, C.B. (1978) "Effects of Uncertain and Nonstationary Parameters upon Capital Market Equilibrium Conditions," J.F.Q.A., Vol. XIII, pp. 419 - 433.

Barry, C.B. and Winkler R.L. (1976) "Non Stationarity and Portfolio Choice," J.F.Q.A., Vol. XI, pp. 217 - 235.

Bartlett, M.S. (1950) "Tests of Significance in Factor Analysis," B.J.P.S.S., Vol.3, pp. 77 -85.

Becker, J.D. (1977) "The Three-Parameter Security Valuation Model : Theoretical and Empirical Considerations," Ph.D Dissertation, University of Washington.

Belkaoui, A. (1977) "Canadian Evidence of Heteroscedasticity in the Market Model," J.F., Vol.XXXII, pp. 1320 - 1323.

Bey, R.P. and Pinches, G.E. (1980) "Additonal Evidence of Heteroscedasticity in the Market Model," J.F.Q.A., Vol. XV, pp. 299 - 322.

Black, F. (1972) "Capital Market Equilibrium with Restricted Borrowing," J.B., Vol. 45, pp. 444 - 455.

Black, F., Jensen, M.C. and Scholes, M. (1972) "The Capital Asset Pricing Model : Some Empirical Tests," in Jensen, M.C., ed., Studies in the Theory of Capital Markets. Praeger Publishers, Inc., New York.

Blalock, H.M. (1972) "Social Statistics," 2nd. ed., McGraw-Hill Book Company, New York.

Blume, M.E. (1970) "Portfolio Theory : A Step Towards its Practical Application," J.B., Vol.43, pp. 153 - 173.

———— (1971) "On the Assessment of Risk," J.F., Vol.XXVI, pp. 1 - 10.

———— (1975) "Betas and their Regression Tendencies," J.F., Vol. XXX, pp. 785 - 796.

Blume, M.E. and Friend, I. (1973) "A New Look at the Capital Asset Pricing Model," J.F., Vol. XXVIII, pp. 19 - 34.

Brealey, R.A. (1979) "The Distribution and Independence of Successive Rates of Return from the British Equity Market," in Richard,P.H., ed. U.K. and European Share Price Behaviour : The Evidence, Kogan Page, London.

Brenner, M. and Smidt, S.(1977) "A Simple Model for Nonstationarity of Systematic Risk," J.F., Vol. XXXII, pp. 1081 - 1092.

Brown, S.J. (1977) "Heteroscedasticity in the Market Model : A Comment," J.B., Vol. 50, pp. 80 - 83.

Chen, N.F. (1981) "Arbitrage Asses Pricing : Theory and Evidence," Ph.D Dissertation, University of California, Los Angeles.

———— (1982) "Some Empirical Tests of the Thoery of Arbitrage Pricing," Working Paper, No. 69, Graduate School of Business, University of Chicago.

Chen, N.F. and Ingersoll, J.E. (1982) "Exact Pricing and Linear Factor Models with Finitely Many Assets," Working Paper, No.74, Graduate School of Business, University of Chicago.

Chen, S.N. (1981a) "A Randam Coefficient Model for Reexamining Risk-Decomposition Method and Risk-Return-Relationship Test," Q.R.E.B., Vol. 20, pp. 60 - 69.

——— (1981b) "Residual Variance Heteroscedasticity Portfolio Diversification and Trading Rules," R.E.B., Vol. 21, pp. 87 - 97.

——— (1981c) "Beta Nonstationarity Portfolio Residual Risk and Diversification," J.F.Q.A., Vol. XVI, pp. 95 - 111.

Chen, S.N. and Keown, A.J. (1981a) " Risk Decomposition and Portfolio Diversification when Beta is Nonstationary : A Note,"J.F., Vol. XXXVI, pp.941 - 947.

——— (1981b) "An Examination of the Relationship Between Pure Residual and Market Risk : A Note," J.F., Vol. XXXVI, pp. 1203 - 1209.

Cohen, K., and Pogue, J. (1967) "An Empirical Evaluation of Alternative Portfolio Selection Models," J.B., Vol. 46, pp. 167 - 193.

Copeland, T.E. and Weston, J.F. (1983) "Financial Theory and Corporate Policy," 2nd. ed., Addison-Wesley Publishing Company, Inc., Reading, Massachusetts.

David, H.A., Hartley, H.O., and Pearson, E.S. (1954) "The Distribution of the ~~Ratio~~, in a Single Normal Sample, of Range to Standard Deviation," B., Vol. 41, pp. 482 - 493.

Dhrymes, P.J., Friend, I. and Gultekin, N.B. (1982) "A Critical Reexamination of the Empirical Evidence on the Arbitrage Pricing Model," Working Paper, No. 12 - 82, the Wharton School, University of Pennsylvania.

Diacogiannis, G.P. (1979) "On Efficient Set Mathematics and Return-Beta Linearity," Ms.c Thesis by Research, University of Manchester Institute of Sciences and Technology.

Douglas, G.W. (1968) "Risk in the Equity Markets : An Empirical Appraisal of Market Efficiency," Ann Arbor, Michigan: University Microfilms, Inc.

Elton, E.J. and Gruber, M.J. (1973) "Estimating the Dependence of Share Prices-Implications for Portfolio Selection," J.F., Vol. XXVIII, pp. 1203 - 1232.

——— (1981) "Modern Portfolio Theory and Investment Analysis," John Wiley and Sons, New York.

Fabozzi, F.J. and Francis, J.C. (1978) "Beta as a Random Coefficient," J.F.Q.A., Vol. XIII, pp. 101- 116.

Fama, E.F. (1965) "The Behaviour of Stock Market Prices," J.B., Vol. 38, pp. 34 - 105.

——— (1968) "Risk, Return and Equilibrium : Some Clarifying Comments," J.F., Vol. XXIII, pp.29 - 40.

——— (1973) "A Note on the Market Model and the two Parameter Model," J.F., Vol.XXVIII, pp.1181 - 1185.

——— (1976) "Foundations of Finance," Basic Books, New York.

——— (1977) "Risk-Adjusted Discount Rates and Capital Budgeting under Uncertainty," J.F.E., Vol. 5,pp.3 - 24.

Fama, E.F. and MacBeth, J.D. (1973) "Risk, Return and Equilibrium : Empirical Tests," J.P.E., Vol. LXXXI, pp. 607 - 636.

——— (1974) "Long-Term Growth in a Short-Term Market," J.F., Vol. XXIX, pp. 857 - 885.

Fama, E.F. and Miller, M.H. (1972) "The Theory of Finance," Dryden Press, Hinsdale, Illinois.

Farrar, D.E. (1967) "The Investment Decision Under Uncertainty," Markham Publishing Company, Chicago.

Farrel, J.L. (1974) "Analyzing Covariance of Returns to Determine Homogeneous Stock Groupings," J.B., Vol. 47, pp. 186 - 207.

Fertuck, L. (1975) " A Test of Industry Indices Based on SIC Codes," J.F.Q.A., Vol. X, pp. 837 - 848.

Fogler, H.R., and Ganapathy, S. (1982) "Financial Econometrics for Researchers in Finance and Accounting," Prentice, Hall, Inc., New Jersey.

Friend, I. (1981) "Discussion," J.F., Vol.XXXVI, pp.350 - 352.

Gehr, A. (1978) "Some Tests of the Arbitrage Pricing Theory," J.M.F.A., Vol. 7, pp. 91 - 105.

Gibbons, M.R. (1981) "Empirical Examination of the Return Generating Process of the Arbitrage Pricing Theory," Working Paper, Stanford University.

Harman, H.H. (1967) "Modern Factor Analysis," The University of Chicago Press, Chicago.

Huberman, G. (1981) "A Simple Approach to Arbitrage Pricing Theory," Working Paper, No.44, University of Chicago.

Hughes, P. (1982) "A Test of the Arbitrage Pricing Theory," Working Paper, University of British Columbia.

Jacob, N.L. (1971) "The Measurment of Systematic Risk for Securities and Portfolios : Some Empirical Results," J.F.Q.A. Vol. VI, pp. 815 - 833.

Jennrich, R.L. (1970) "An Asympotic x^2 Test for the Equality of two Correlation Matrices," J.A.S.A., Vol. 65, pp. 904 - 912.

Johnson, K.J. (1981) "An Orthogonal Factor Model for Pricing Capital Assest," Working Paper, Department of Management Sciences, University of Manchester Institute of Science and Technology.

Jöreskog, K.G. (1963) "Testing a Simple Structure Hypothesis in Factor Analysis," P., Vol. 31, pp. 165 - 178.

Jöreskog, K.G. and Sörbon, D. (1979) "Advances in Factor Analysis and Structural Equation Models," Abt. Book Inc., Cambridge, Massachusetts.

Kendall, M.G. and Buckland, W.R. (1978) "A Dictionary of Statistical Terms," 3rd.ed., Published by Oliver and Bond, Edinburgh.

King, B.F. (1966) "Market and Industry Factors in Stock Price Behaviour," J.B., Vol. 39, pp. 139 - 190.

Kraus, A and Litzenberger, R. (1976) " Skewness Preference and the Valuation of Risk Assets," J.F., Vol.XXXI, pp. 1085 - 1100.

Kryzanowski, L. and Chau, T.M. (1982) "General Factor Models and the Structure of Security Returns," Working Paper, University of Concordia, Montreal.

Langetieg, T.C. (1978) "An Application of a Three-Factor Performance Index to Measure Stockholders Gains from Merger," J.F.E., Vol. 6, pp. 365 - 383.

Lee, G.F. and Chen, S.N. (1980) "A Random Coefficient Model for Reexamining Risk-Decomposition Method and Risk Return Relationship Test," Q.R.E.B., Vol. 20, pp. 58 - 65.

Levy, H. (1978) "Equilibrium in an Imperfect Market : A Constraint on the Number of Securities in the Portfolio," A.E.R., Vol.38, pp. 643 - 658.

Levy, R.A. (1971) "On Short Term Stationarity of Beta Coefficients," F.A.J., Vol.27, pp. 55 - 62.

Lintner, J. (1965) " The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budget," R.E.S., Vol. XLVII, pp.13 - 37.

Livingston, M. (1977) "Industry Movements of Common Stocks," J.F., Vol. XXXII, pp. 861 - 874.

Litzenberger, R.H. and Ramawsamy, K. (1979) "The Effect of Personal Taxes and Dividends on Capital Asset Prices," J.F.E., Vol. 7, pp. 163 - 195.

Lawley, D.N. and Maxwell, A.E. (1977) "Factor Analysis as a Statistical Method," 2nd. ed., Butterworths and Co., Publishers Ltd., London.

Mandelbrot, B. (1963) "The Variation of Certain Speculative Prices," J.B., Vol. 36, pp. 394 - 419.

Mardla, K.V. (1971) "The Effect of Nonnormality on Some Multivariate Tests and the Robustness to Nonnormality in the Linear Models," B.,Vol. 58, pp. 105 - 121.

Markowitz, H. (1952) "Portfolio Selection," J.F., Vol.VVI, pp. 77 - 91.

——————(1959) "Portfolio Selection," John Wiley and Sons, New York.

Mayers, D. (1972) "Nonmarketable Assets and Capital Market Equilibrium under Uncertainty," in Jensen, M.C., ed., Studies in the Theory of Capital Markets, Praeger Publishers, Inc., New York.

—————— (1973) "Nonmarketable Assets and the Determination of Capital Asset Prices in the Absence of a Riskless Asset," J.B., Vol. 46, pp. 258 - 267.

Mayers, S.L. (1973) "A Reexamination of Market and Industry Factors in Stock Price Behaviour," J.F., Vol.VIII, pp. 695 - 705.

Merton, R.C. (1972) "An Analytic Derivation of the Efficient Portfolio Frontier," J.F.Q.A., Vol. VII, pp. 1851 - 1872.

Miller, M. and Scholes, M. (1972) "Rates of Return in Relation to Risk : A Reexamination of Some Recent Findings," in Jensen, M.C., ed., Studies in the Theory of Capital Markets, Praeger Publishers, Inc., New York.

Morrison, D.F. (1976) "Multivariate Statistical Methods,"2nd. ed., McGraw-Hill Book Company, New York.

Mossin, J. (1966) "Equilibrium in a Capital Asset Market," E., Vol. 34, pp. 768 - 783.

Milne, F. and Smith, C.F. (1980) "Capital Asset Pricing with Proportional Transaction Costs," J.F.Q.A., Vol. XV, pp. 253 - 266.

Nering, E.D. (1970) "Linear Algebra and Matrix Theory," John Wiley and Sons, New York.

Officer, R.R. (1972) "The Distribution of Stock Returns," J.A.S.A., Vol. 67, pp. 807 - 812.

Ohlson, J.A. and Garman, M.B. (1980) "A Dynamic Equilibrium for the Ross Arbitrage Model," J.F., Vol. XXXV, pp. 675 - 684.

Peasnell, K. and Skerratt, L.C.L. and Taylor, P.A. (1979) "An Arbitrage Rational for Tests of Mutual Fund Performance," J.B.F.A., Vol. 6, pp. 373 - 400.

Rao, C.R. (1955) "Estimation and Tests of Significance in Factor Analysis," P., Vol. 20, pp. 93 - 111.

Reinganum, M.R. (1980) "A Simple Test of the Arbitrage Pricing Theory," Working Paper, Graduate School of Business, University of Southern California.

——— (1981) "The Arbitrage Pricing Theory : Some Empirical Results," J.F., Vol. XXXVI, pp. 313 - 321.

Roll, R. (1969) "Bias in Fitting the Sharpe Model to Time Series Data," J.F.Q.A., Vol. III, pp. 274 - 281.

——— (1977) "A Critique of the Asset Pricing Theory's Tests : Part I : On the Past and Potential Testability of the Theory," J.F.E., Vol.4, pp. 129 - 176.

——— (1978) "Ambiguity when Performance is Measured by the Securities Market Line," J.F., Vol. XXXIII, pp. 1051 - 1069.

——— (1979) "Testing a Portfolio for Ex Ante Mean/Variance Efficiency," in Elton, E. and Gruber, M., eds., TIMS Studies in the Management Sciences, Amsterdam, North Holland.

Roll, R. and Ross, S.A. (1980) "An Empirical Investigation of the Arbitrage Pricing Theory," J.F., Vol. XXXI, pp. 1073 - 1103.

Rosenberg, B. and Ohlson, J.A. (1976) "The Stationary Distribution of Returns and Portfolio Separation in Capital Markets : A Fundamental Contradiction," J.F.Q.A., pp. 393 - 401.

Ross, S.A. (1976) "The Arbitrage Theory of Capital Asset Pricing," J.E.T., Vol. 13, pp. 341 - 360.

——— (1977) "Return, Risk, and Arbitrage," In Friend, I. and Bickskler, J., ed., Ballinger Cambridge, Massachusetts.

Rummell, R.J. (1970) " Applied Factor Analysis," Evanston, Ill : Northwestern University Press.

Schalheim, J. and DeMagistris, R. (1980) "New Estimates of the Market Parameters," F.M., Vol. 9, pp. 60 - 68.

Schwartz, R.A. and Whitcomb, D.K. (1977) "Evidence on the Presence of Causes of Serial Correlation in Market Model Residuals," J.F.Q.A., Vol.XII, pp. 291 - 313.

Shanken, J. (1982) "The Arbitrage Pricing Theory : Is It Testable? " J.F., Vol. XXXVII, pp. 1129 - 1140.

Sharpe, W.F. (1963) "A Simplified Model for Portfolio Analysis," M.S., Vol. 9, pp. 227 - 293.

——— (1964) "Capital Asset Prices : A Theory of Market Equilibrium under Conditions of Risk," J.F., Vol. XIX, pp. 425 - 442.

——— (1977) "The Capital Asset Pricing Model : "A Multi-Beta" Interpretation," In Levy H. and Sarnat, M., eds., Financial Decision Making Under Uncertainty , Academic Press, New York.

Stapleton, R.C. and Subrahmannam, M.G. (1978) "Market Imperfections, Uncertain Information and Capital Market Equilibrium," In Sarnat in ed. Inflation and Capital Markets, Ballinger, Cambridge, Massachusetts.

Sunder, S. (1980) "Stationarity of Market Risk : Random Coefficients Tests for Individual Stocks," J.F., Vol. XXXV, pp. 883 - 896.

Swamy, P.A.J.B. (1970) "Efficient Inference in a Random Coefficient Regression Model," E. Vol. 38, pp.311 - 323.

Teichmoeller, D. (1971) " A Note on the Distribution of Stock Price Changes," J.A.S.A., Vol. 66, pp. 272 - 284.

Theil, H. and Mennes, L.B.M. (1959) "Multiplicative Randomness in Time Series Regression Analysis," Mimeographed Report No. 5901 of the Econometric Institute of the Netherlands School of Economics.

Theobald, M. (1979) " A Note on Variable Transactions Costs, the Capital Asset Pricing Model and the Corporate Dividend Decision," J.B.F.A., Vol. 6, pp. 9 - 16.

——— (1980) " An Analysis of the Market Model and Beta Factors using U.K. Equity Share Data," J.B.F.A., Vol. 7, pp. 49 - 64.

Wallingford, B.A. (1967) "A Survey and Comparison of Portfolios Selection Models," J.F.Q.A., Vol. II, pp. 85 - 106.

Westerfield, R. (1977) "The Distribution of Common Stock Price Changes : An Application of Transactions Time and Subordinated Stochastic Models," J.F.Q.A., Vol. XII, pp. 743 - 765.

Weston, J.F. (1981) "Developments in Finance Theory," F.M., Vol. 10, pp. 5 - 22.